FEATURE ARTICLES

Mathematics Learning in Language Inclusive Classrooms: Supporting the Achievement of English Learners and Their English Proficient Peers

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National and state assessments show that English language learners (ELs) in elementary and secondary grades score lower in mathematics compared with their matched English proficient peers (EPs). To provide information on strategies for enhancing learning opportunities for ELs in language inclusive classrooms, we analyze the efficacy of *Learning Mathematics Through Representations* (LMR), a curriculum unit on integers and fractions designed to support learning opportunities for ELs as well as EPs. LMR features the number line as a principal representational context and the use of embodied representations to support students as they explore mathematical ideas, construct arguments, and elaborate explanations. The study used a quasi-experimental design: Twenty-one elementary classrooms employing a highly regarded curriculum were included. Forty-four ELs were enrolled across 11 LMR classrooms, and 51 ELs were enrolled across 10 matched comparison classrooms. Multilevel analysis of longitudinal data on a specialized integers and fractions assessment, as well as a California state mathematics assessment, revealed that the ELs in LMR classrooms showed greater gains than comparison ELs and gained at similar rates to their EP peers in LMR classrooms. Further, contrasts between ELs in the LMR classrooms and EPs in the comparison classrooms revealed that LMR narrowed or eliminated the pretest achievement gap in mathematics. Both theory and empirical results support the value of LMR as a mathematics intervention benefitting both EL and EP students.

Keywords: achievement gap; assessment; curriculum; diversity; elementary mathematics education; English language learners; evaluation; item response theory; learning environments; mixed methods; outcome measures; quasi-experimental analysis

Students classified as English learners (ELs) show lower test scores in mathematics relative to English proficient students (EPs) at fourth and eighth grades on both national assessments and state assessments (Carnoy & García, 2017; Hemphill & Vanneman, 2011). The EL-EP achievement gap points to persistent inequities in mathematics learning opportunities for ELs, and educators are only beginning to understand how to address concerns about differential opportunities (Goldenberg, 2013; Janzen, 2008; Jensen, 2017; Hakuta & Santos, 2012; Moschkovich, 2012, 2013). This article reports on the efficacy of a 19-lesson experimental curriculum unit about hard-to-learn and hard-to-teach ideas in the domains of integers and fractions in language inclusive classrooms. The unit, *Learning Mathematics Through Representations* (LMR), supports learning opportunities for EL and EP students through the use of the number line as a principal representation to explore mathematical ideas, construct arguments, and elaborate explanations in classroom discussions and individual work (Gearhart & Saxe, 2014; Saxe, de Kirby, Kang, Le, & Schneider, 2015).

In a previous publication, Saxe, Diakow, and Gearhart (2013) reported evidence of LMR's efficacy based on comparisons between the achievement gains of LMR students versus students in a matched comparison group. Saxe and colleagues recruited teachers from three urban districts that employed a well-regarded district curriculum (*Everyday Mathematics*, [University of

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Educational Researcher, Vol. 48 No. 7, pp. 452–465
DOI: 10.3102/0013189X19869953
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Chicago School Mathematics Project, 2007]). All participating teachers were told that they would have access to LMR lessons and professional support either during the study (LMR group) or during the subsequent academic year (comparison group). To create the LMR group ($n = 11$) and a matched comparison group ($n = 10$), Saxe and colleagues matched teachers on three indicators: greatest terminal degree, years of teaching experience, and previous professional development. Teachers were then assigned to groups with care taken not to assign teachers from both LMR and comparison groups to the same school to minimize the risk that comparison teachers might implement any of the LMR lessons. Teachers assigned to the LMR group then participated in professional development sessions to support lesson implementation, and they were asked to implement LMR during their allotted time for mathematics instruction.

To document student learning related to integers and fractions, Saxe and colleagues (2013) developed a specialized assessment with items adapted from *Everyday Mathematics*, LMR lesson materials, released items from the National Assessment of Educational Progress (NAEP), and released items from California's testing program. In the assessments, authors balanced items that involved number line representations with other representations (e.g., numbers only, area models). The specialized assessment was administered on four occasions: pretest in September, interim test in October (LMR only), posttest in December, and final test in May.

As Saxe and colleagues (2013) have previously reported, to estimate student achievement on four integers and fractions assessments, multidimensional item response models (Adams, Wilson, & Wang, 1997) were used. The models enabled the researchers to link estimates of student achievement on pre, interim, post, and final assessments, based on performance on 18 common items and 11 to 14 additional items that shifted in emphasis from integers to fractions over the assessments. Importantly, the item response model, scaled as logits, creates an interval-level measurement scale from ordinal or categorical items (e.g., correct/incorrect responses). Complementing the item response modeling, longitudinal growth modeling enabled Saxe and colleagues to estimate changes in achievement over time. Growth modeling, a form of multilevel modeling, was selected because students were clustered within classrooms and their growth was measured during the school year with pre, interim, post, and final assessments.

In this paper, we disaggregate the prior analyses to focus on the achievements of ELs as contrasted with those of EPs as reflected in the integers and fractions assessments. In addition, we examine student performance on a California state standardized assessment (California Standardized Test [CST]). The standardized assessment data, not included in prior reports, allows us to evaluate broader effects of the LMR intervention on both EP and EL students' general mathematics achievement.

Learning Mathematics Through Representations: *Research Base and Expected Support for Both English Learners and English Proficient Students*

The product of design-based research (Saxe, de Kirby, Le, Sitabkhan, & Kang, 2015), LMR is a 19-lesson curriculum unit on integers and fractions. The cross-lesson use of the number line provides continuity of ideas and supports students' efforts to build on prior insights in subsequent lessons. As depicted in Figure 1, in the early integers lessons, students engage with activities and discussions about positive integers as they create, define, and reflect upon units and multiunits to the right of zero on the number line; in later integers lessons, students extend the ideas of unit and multiunit to numbers to the left of zero (negative integers). Similarly, as depicted in Figure 2, the fractions lessons begin with the idea of fractions as splitting integers into subunits on the number line. Later lessons advance to multiplicative relations between fraction numerators and denominators.

Each LMR lesson consists of a five-phase structure that supports teachers' efforts to build upon student thinking in instructional activities. As depicted in Figure 3, lessons begin with two or three nonroutine *opening problems* that introduce lesson content, serve as formative assessments of students' diverse conceptual understandings, and provide a focus for the *opening discussion*. The *opening problem* featured at the center of Figure 3, for example, presents a number line with only the numbers 6 and 7 labeled, and students are asked to label a third number at the leftmost position on the line. Students' responses often reveal two common ideas: that numbers should be ordered consecutively (5, 6, 7) on the line, leading to "5," not "4," as the solution to the problem in Figure 3; and the idea that the leftmost position on the line should be "0." In the *opening discussion*, when students explain their thinking about the *opening problems*, the teacher introduces or reviews a mathematical principle regarding the number line as well as actions on the line to support the resolution of conflicting ideas. For the lesson illustrated in Figure 3, the teacher (a) introduces the definitions of *interval* as "the distance between any two numbers on the number line" and *unit interval* as "the distance from 0 to 1 or any distance of 1," and (b) encourages actions on the line such as displacing a unit interval from one position to another with Cuisenaire rods or pinched fingers. During *partner work*, students apply insights from the *opening discussion* as they solve problems that are sequenced in difficulty. In the *closing discussion*, the teacher encourages students to communicate ideas and guides the class to resolve disagreements. The lesson concludes with *closing problems* that provide teachers with an assessment of student thinking and progress.

In the LMR curriculum, the recurrent use of the five-phase lesson structure supports all students in mathematical discussion, argumentation, and problem solving. Additional supports for argumentation and problem solving that may be particularly useful for EL (as well as EP) students include (a) supports for teachers' (and students') coordinated use of visual, embodied, verbal, physical (manipulative), and written representations, as well as (b) supports for teachers' cultivation of classroom norms that encourage explanation and careful listening to fellow students' contributions. We note below the warrants for such design features of lessons in the mathematics education research literature.

Mathematical discussion, argumentation, and problem solving. Many mathematics educators argue that K–12 mathematics education should emphasize argumentation and problem solving (e.g., National Council of Teachers of Mathematics

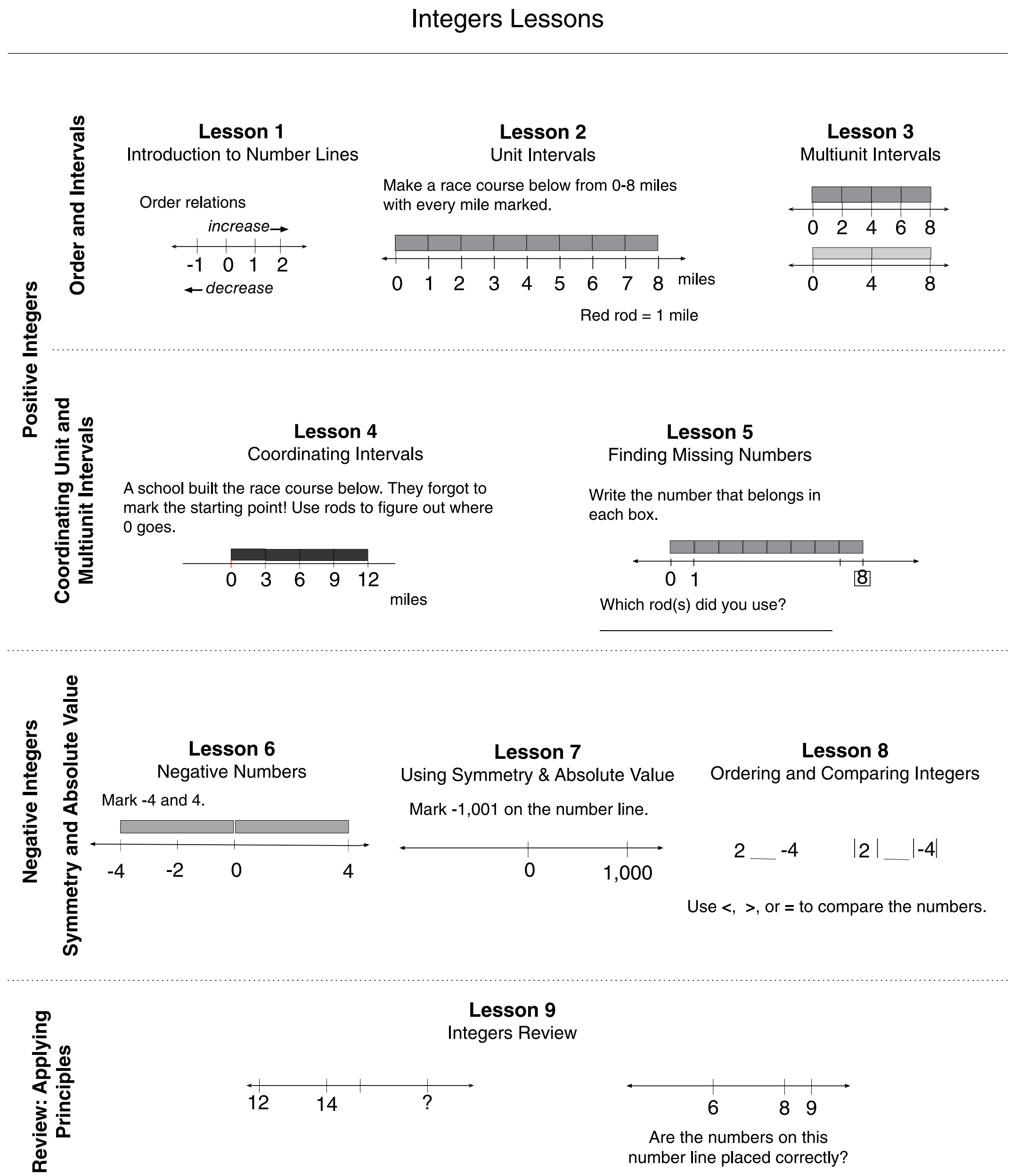

FIGURE 1. *Integers lessons in the* Learning Mathematics Through Representations *sequence.*

[NCTM], 2000; National Governors Association Center for Best Practices [NGACBP] & Council of Chief State School Officers [CCSSO], 2010; Schoenfeld, 2002), and their arguments are consistent with established theoretical treatments of cognitive development and mathematics learning (e.g., Piaget, 1970; Sfard, 2008; Vygotsky, 1986). The basic idea is that all students, including ELs, develop mathematical ability through participation in discourse practices, such as presenting arguments, responding to the arguments of others, and explaining solutions (Moschkovich, 2002). Such an emphasis on active participation in discursive practices is a marked departure from common practices in classrooms serving ELs, especially low-income ELs of Latinx descent; practices in these classrooms often emphasize learning lower-level skills, such as computation, rote memorization of facts, and finding key phrases in word problems (Darling-Hammond, 2007; Khisty & Viego, 1999; Moschkovich, 2002; Secada, Ortiz-Franco, Hernandez, & De La Cruz, 1999). In contrast, the design features of LMR's five-phase recurring lesson structure is consistent with current views about the import of student participation in argumentation, discussion, and problem solving.

Supports for teachers' (and students') coordinated use of visual, embodied, verbal, physical (manipulative), and written representations. Many education scholars agree that high quality mathematics lessons should encourage students to use and coordinate resources, such as visual/physical materials, embodied actions, and linguistic representations, to support communication and reflection (Hakuta & Santos, 2012; Moschkovich, 2002; Schleppegrell, 2007). Consistent with these views, LMR lessons engage students with visual and physical representations through use of the number line and Cuisenaire™ rods to represent distances on

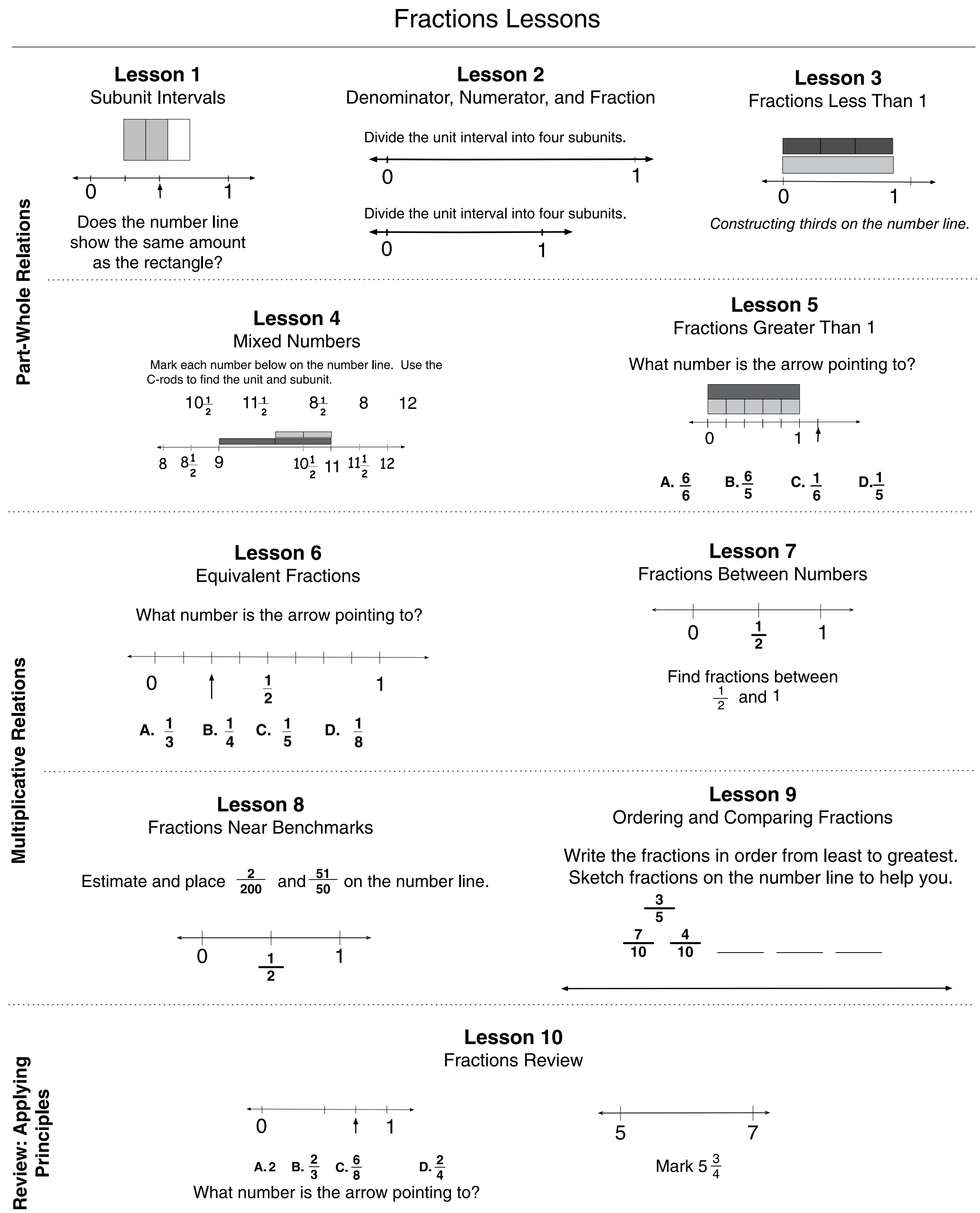

FIGURE 2. *Fractions lessons in the* Learning Mathematics Through Representations *sequence.*

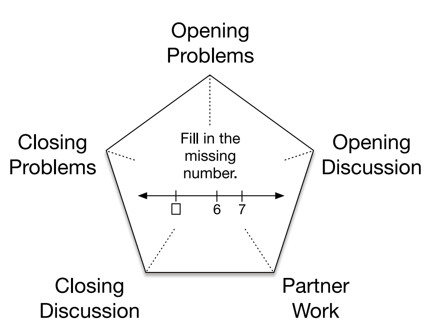

FIGURE 3. *Five-phase* Learning Mathematics Through Representations *lesson structure with an example of a nonroutine problem.*

the number line. In early integers lessons, for example, students initially use Cuisenaire rods to measure distances on open number lines with only 0 labeled (e.g., locating the integer 3 as a distance of three red rods from 0). In so doing, students engage in the actions of placing, iterating, and partitioning rods and intervals as they work through the challenges of quantifying continuous linear distances. Students' actions, and their reflections on their actions, may be particularly useful for ELs' mathematical development (Bustamante & Travis, 1999; see also Piaget, 1970 for the role of actions in his constructivist treatment). In later integers problems, the number line is labeled with more than one point (e.g., 0 and 1, or 2 and 4), and the rods become means for students to measure the distance between labeled points and then locate additional values along the line using the rods. In advanced fractions lessons, students investigate ideas like equivalent fractions by using rods to split a marked unit interval into "subunit" intervals of different lengths (e.g., two subunits for halves, four subunits for fourths).

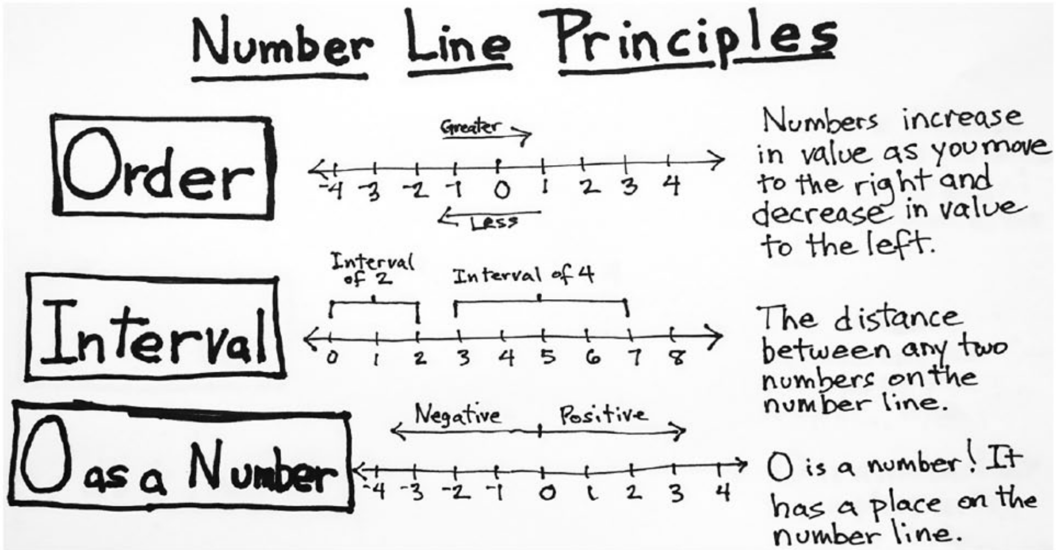

FIGURE 4. *Classroom poster showing three of the* Learning Mathematics Through Representations *principles/definitions and their graphical representations.*

Additionally, linguistic representations (oral and written), coordinated with visual and embodied representational forms, are important features of LMR lessons. For example, LMR's core mathematical vocabulary, termed "number line principles and definitions," are progressively recorded on a classroom poster. Over the course of the lesson sequence, the poster provides students access to number line definitions and principles for foundational ideas, like unit, multiunit, and subunit. Figure 4 contains illustrative definitions/principles introduced early in the integers lessons for order, interval, and zero as a number. LMR's lesson guides encourage teachers to create opportunities for students to use definitions to resolve conflicts and to support argumentation. One recommended technique is "pushing student thinking" to engage students in correcting the incorrect reasoning of a hypothetical person by justifying their argument with reference to number line principles and definitions (Saxe, de Kirby, Kang, et al., 2015). Online supplement #1 (S1, available on the journal website) contains the entire set of definitions/principles developed over the course of integers and fractions lessons, and online supplement #2 (S2, available on the journal website) contains an introduction to the lesson guides and the use of definitions/principles in the lessons.

Productive norms and routines. Classroom norms that value participation and argumentation are regarded by many educators as a key feature of high quality mathematics instruction (Doty, Mercer, & Henningsen, 1999; Hiebert et al., 1996; Yackel & Cobb, 1996). Ramirez and Bernard (1999) suggest that ELs' mathematical learning opportunities suffer in lecture-based and textbook-centered classrooms (see also Warren, Quine, and DeVries [2012] and Fuson, Smith, and Lo Cicero [1997]). In contrast to lecture-based/textbook-centered classrooms, LMR classrooms support the norms in which there is an expectation that students will (a) reference definitions to support argumentation, (b) offer conjectures and explanations during classroom discussion, and (c) listen carefully to their peers' ideas in discussions and partner work. The participation structures provided by LMR and inquiry practices that they support are quite different from what has been reported as typical for ELs.

The Current Study

Previously reported findings provided evidence for the efficacy of LMR lessons using our specialized measure of integers and fractions achievement (from Saxe, Diakow, et al., 2013), but to date, Saxe and colleagues have not reported analyses of differential achievement gains for EP versus EL students, nor reported findings on students' performance on a standardized assessment of general mathematics achievement. The features of LMR are well aligned with what some educators recommend as best practices for EL students, and we expected that LMR affords learning opportunities for ELs beyond the standard integers and fractions curriculum. We therefore conducted new analyses with the expectation that the findings would reveal the following:

(1) ELs who participate in LMR will show greater gains in mathematics than ELs in comparison classrooms.
(2) ELs and EPs who participate in LMR will show similar gains in mathematics (i.e., no detectable difference in rates of learning as a function of language status and thus similar learning opportunities).
(3) The post-intervention achievement gap between ELs who participate in LMR classrooms and EPs who participate in comparison classrooms will be reduced relative to their pre-intervention gap.
(4) ELs (and EPs) who participate in LMR will develop integers and fractions knowledge not solely linked to the number line; they will show progress on items for which there are no number line representations.

Method

Participants

The participants included 571 fourth and fifth grade students from three urban and suburban school districts in the San Francisco Bay Area. The students reflected the ethnic/racial and socioeconomic diversity of the urban school districts in

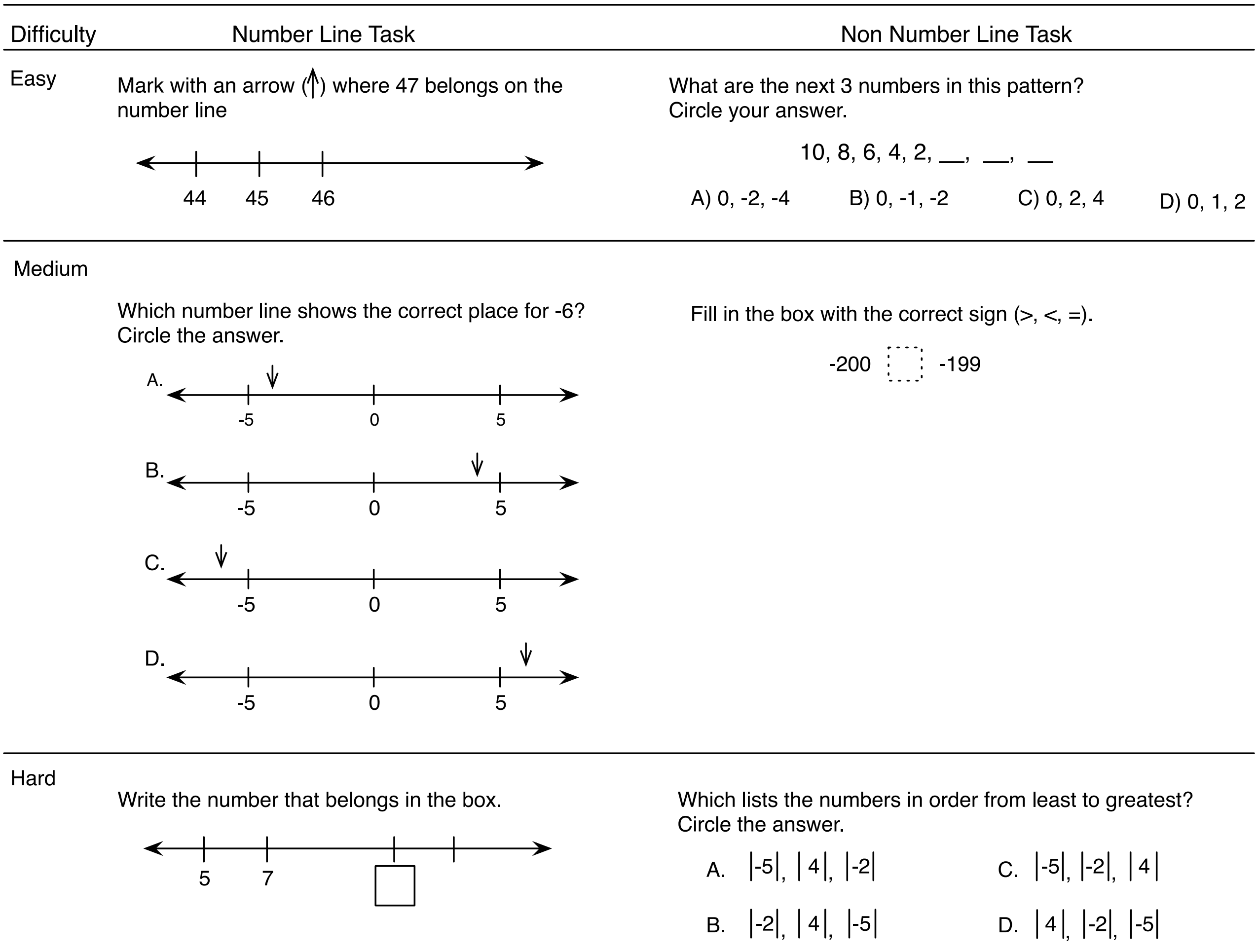

FIGURE 5. *Sample integers assessment items with and without number line representations integers at three difficulty levels.*

which this study was conducted. Forty-four ELs participated in LMR classrooms, and 51 ELs participated in comparison classrooms. All classrooms contained participants classified as ELs. Chi-square tests showed no statistical difference between LMR and comparison ELs for numbers of EL students and their sex or ethnicity. The ethnic distribution of the sample included students who identified as Latinx, Pacific-Asian, Multi-ethnic, African American, and White. Online supplement #3 (S3, available on the journal website) contains (a) additional details on the ethnic distribution and other characteristics of the sample, and (b) statistical analyses that examined the relationship between ELs' ethnicities and the effects of the LMR intervention. The analyses found no evidence of an association between ELs' ethnicity and the impact of LMR.

Implementation and Fidelity of Implementation

LMR teachers participated in a three-day professional development summer institute, followed by four evening meetings during implementation. Analyses of multiple sources confirmed that LMR teachers implemented all 19 LMR lessons (sources included self-report teacher surveys, student work, video recordings [Gearhart & Saxe, 2014; Saxe, Diakow, et al., 2013; Saxe, de Kirby, Le, et al., 2015]). Supplement #4 (S4, available on the journal website) contains links to LMR lessons.

Assessments and Data Collection

Students completed a specialized assessment of integers and fractions (Saxe, Diakow, et al., 2013) and the California standardized assessment in mathematics.

The specialized assessment of integers and fractions is a set of three assessments linked through item response theory modeling; the assessments vary in difficulty and content. The assessments contain items adapted from a range of sources, and item formats balance number line representations versus other representations (e.g., numbers only, area models). Sample assessment items are contained in Figures 5 and 6 for integers and fractions, respectively. The figures show that each assessment contains items with number line representations and those without, at a range of difficulty levels (difficulty being related to the proportion of correct responses for each item). The specialized assessment was administered on four occasions—pretest in September, interim test in October (LMR only), posttest in December, and final test (identical to the posttest) in May. A set of 18 common items, used in all three assessments, allowed us to link student scores from the different time points using item response models. Each assessment contained an additional 11–14 unique items that were intended to assess specific forms of learning: The unique items were easier at pretest and harder at final test, and the content emphasis shifted over time from integers to fractions.

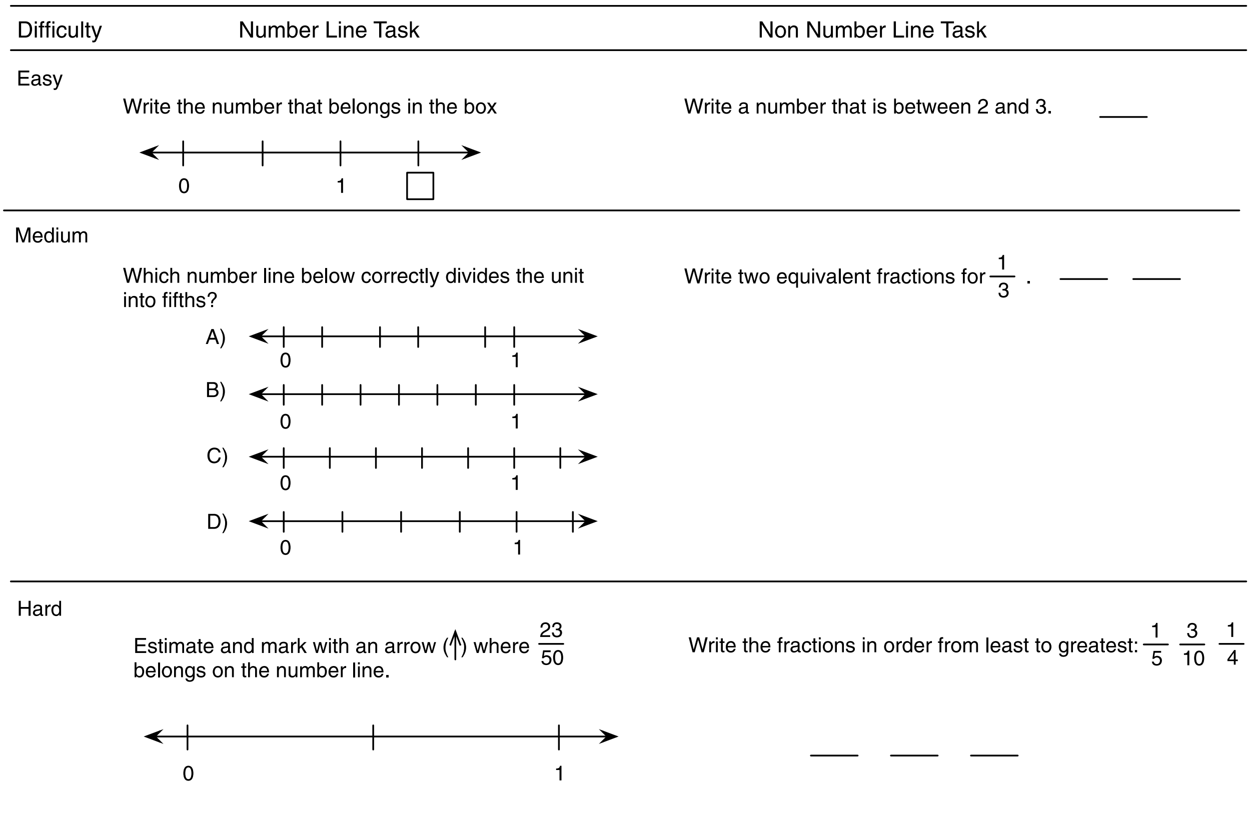

FIGURE 6. *Sample fractions assessment items with and without number line representations integers at three difficulty levels.*

The design of the specialized assessment for measuring integers and fractions achievement supported its validity in several ways. The strong linking design using many items guarded against measurement invariance (e.g., items showing invariance could be dropped); the shift in item difficulty ensured accurate estimates at each time point (e.g., by avoiding floor/ceiling effects and practice effects) and the use of items that did not utilize a single representation or curriculum source supports the validity of treatment group comparisons. Additional validity evidence comes from the relations to other variables: Correlations between the integers and fractions assessments and the standardized assessments ranged from $r = 0.69$ to $r = 0.77$ [$p < .001$ in all cases]). Finally, each test form demonstrated good reliability (Cronbach's alpha was above 0.85 for each form). Online supplement #5 (S5, available on the journal website) contains the complete set of integers and fractions assessment forms for pretest, interim test, and post/final test.

The CST was administered at two points, the end of the previous school year (prior year test) and the end of the school year in which the LMR intervention was implemented (end-of-year test).

Data analysis. To investigate the efficacy of LMR, we conducted multilevel analysis of longitudinal data on the integers and fractions assessment as well as the standardized assessment. As in our prior work (Saxe, Diakow, et al., 2013), to examine evidence of student learning from performance on the specialized assessment, we used multidimensional item response modeling to estimate student achievement at four time points followed by piecewise longitudinal growth modeling to estimate changes in achievement over time. See online supplement #6 (S6) for details on the item response theory measurement model. Online supplement #7 (S7) contains descriptive statistics for the assessment data. Online supplement #8 (S8) contains the statistical models for analyzing the assessment data. (All online supplements are available on the journal website.) We conducted statistical analyses based on the fixed effects from the model and applied an $\alpha = 0.05$ level of significance for all analyses.

Missing data. During the LMR study, some students enrolled in school and joined participating classrooms, while others left participating classrooms. Further, some students were absent during assessments and make-up assessments. As a result, 83% of participants had complete data for the integers and fractions assessment; 72% had complete data for the standardized assessment; 86% had data for ethnicity; and 87% had data on EL status. We investigated the impact of missing data on the internal validity of the study. First, we tested for associations between missing data and assessment scores. The results indicate that the missing data are missing completely at random (e.g., there is no evidence of a correlation between missing variables such as ethnicity or EL status with assessment scores). Second, we compared the estimates from statistical models for all participants whether or not they had complete data records (e.g., data for both assessments) versus including only participants who had complete data (i.e., listwise deletion). The statistical models fit using all participants yielded practically identical results with the statistical models fit using only those cases with complete data (e.g., no changes in statistical significance of parameter estimates). The final models were fit using all available data to maximize information. Supplement #9 (S9, available on the journal website) contains additional details about how we addressed issues related to missing data.

Results

Figure 7 contains a six-panel summary of key results from the integers and fractions assessment (Figure 7a) and the standardized assessment (Figure 7b). The numerical estimates and standard errors for the six-panel summary are contained in online supplements #10a (S10a) for the integers and fractions assessment and online supplement #11a (S11a), both available on the journal website, for the standardized assessment. Due to differences in the way the two assessments were administered, scaled, and analyzed, the findings for the two assessments are represented differently.

Students' performance on the integers and fractions assessments, shown in Figures 7a.i–7a.iii, are represented in logits. The graphs show students' shifting achievement over the four assessments during the academic year in which LMR was implemented: pre, interim (during LMR), post (after LMR), and final (five months after LMR).

Students' performance on the standardized assessments, shown in Figures 7b.i–7b.iii, are represented as scaled scores. The graphs show students' achievement in the year prior to LMR and at the end of the LMR intervention year. The horizontal dashed bars represent one year of expected growth; thus, a student who obtained a score of 435 on the prior year assessment and 435 on end-of-year assessment showed the expected growth of one year.[1] The proficiency level thresholds depicted in the graphs—Basic, Proficient, Advanced—are included to orient the reader. LMR and comparison classrooms included both fourth and fifth grades, and the thresholds (cut scores) for the "advanced" category created by the California Department of Education varied by grade. We represent this ambiguity with the grayed area: Students who achieved scores between 400 and 430 (grayed area) would be categorized either as "proficient" or "advanced" dependent upon their grade level.

In the analyses to follow, we use the findings presented in Figure 7 to examine whether our first three expectations received corroboration for both the integers and fractions assessment and the standardized assessment. We summarize the findings related to all expectations in Table 1, and online supplement #12 (S12, available on the journal website) contains the standardized treatment effects across all measures and times of testing. In addition, Figure 8 contains the results that address our fourth expectation related to LMR—that EL (and EP) students will show strong growth on integers and fractions items that contain number lines as well as those that do not. In the following sections and in the online supplements, we present our analyses and more nuanced examination of our findings related to each expectation.

(1) LMR Efficacy for ELs

To test our first expectation—that ELs who participate in LMR will show greater gains in mathematics than ELs in comparison classrooms—we compared the expected growth of ELs between LMR and comparison groups.

Integers and fractions assessment. We used a piecewise longitudinal growth model to estimate the mean student achievement on the integers and fractions assessment for ELs as a function of treatment group (LMR versus comparison classrooms) at pre, interim, post, and final test.

Figure 7a.i contains a graph of estimated integers and fractions achievement in logits for ELs over the four assessments. The figure reveals different patterns of growth for ELs in LMR and those in comparison classrooms: ELs in LMR classrooms showed steady growth over time, though less growth during the five-month gap between posttest and final test. In contrast, ELs in the comparison classrooms showed less growth from pretest to posttest but sharper growth from posttest to final test. Statistical analysis confirms trends observable in Figure 7a.i. Performance of the two EL groups was comparable at pretest (no statistical difference between groups [$p = .736$]), but ELs in LMR gained an estimated 1.41 ($SE = 0.19$) logits more from pre to post than EPs in the comparison group ($p < .001$), with a standardized effect size (ES) of 0.86 SD.[2] Performance on the final test (administered five months after LMR concluded) shows that LMR ELs maintained their advantage over ELs in comparison classrooms; ELs in LMR gained an estimated 0.79 ($SE = 0.20$) logits more from pre to final than EPs in the comparison group ($p < .001$), with an ES of 0.48 SD. We note that the growth spurt between post and final assessments for ELs in comparison classrooms may be due to the observation that the *Everyday Mathematics* text spent greater time with integers and fractions in the spring. Regardless, comparison ELs' growth did not lead to the level of growth of ELs in LMR classrooms. Online supplement #10b (S10b, available on the journal website) contains the complete regression results for the integers and fractions model.

Standardized assessment. To investigate whether LMR supported ELs' grade-level proficiency as measured by the standardized assessment in math, we analyzed the estimated change in mean scores from the prior-year to the end-of-year assessments; the analysis corroborates the findings produced with the integers and fractions assessment.

Figure 7b.i shows an overall rise in expected score for ELs participating in LMR classrooms and an overall decline for ELs participating in comparison classrooms. The differential change, contrasting the LMR and comparison groups, is substantial: There is a 33.7-point difference between the EL gain in the LMR group and the EL decline in the comparison group ($p = .013$, $ES = 0.37$), which is equivalent to nearly a year's growth in mathematics achievement at the upper elementary grades (Lipsey et al., 2012). Online supplement #11b (S11b, available on the journal website) contains the complete regression results for the standardized assessment model.

(2) LMR Efficacy for ELs Versus EPs

To test our second expectation—that LMR supports similar rates of learning for both EL and EP students—we compared the average growth of ELs who participated in LMR classrooms with that of EP students who participated in the same LMR classrooms.

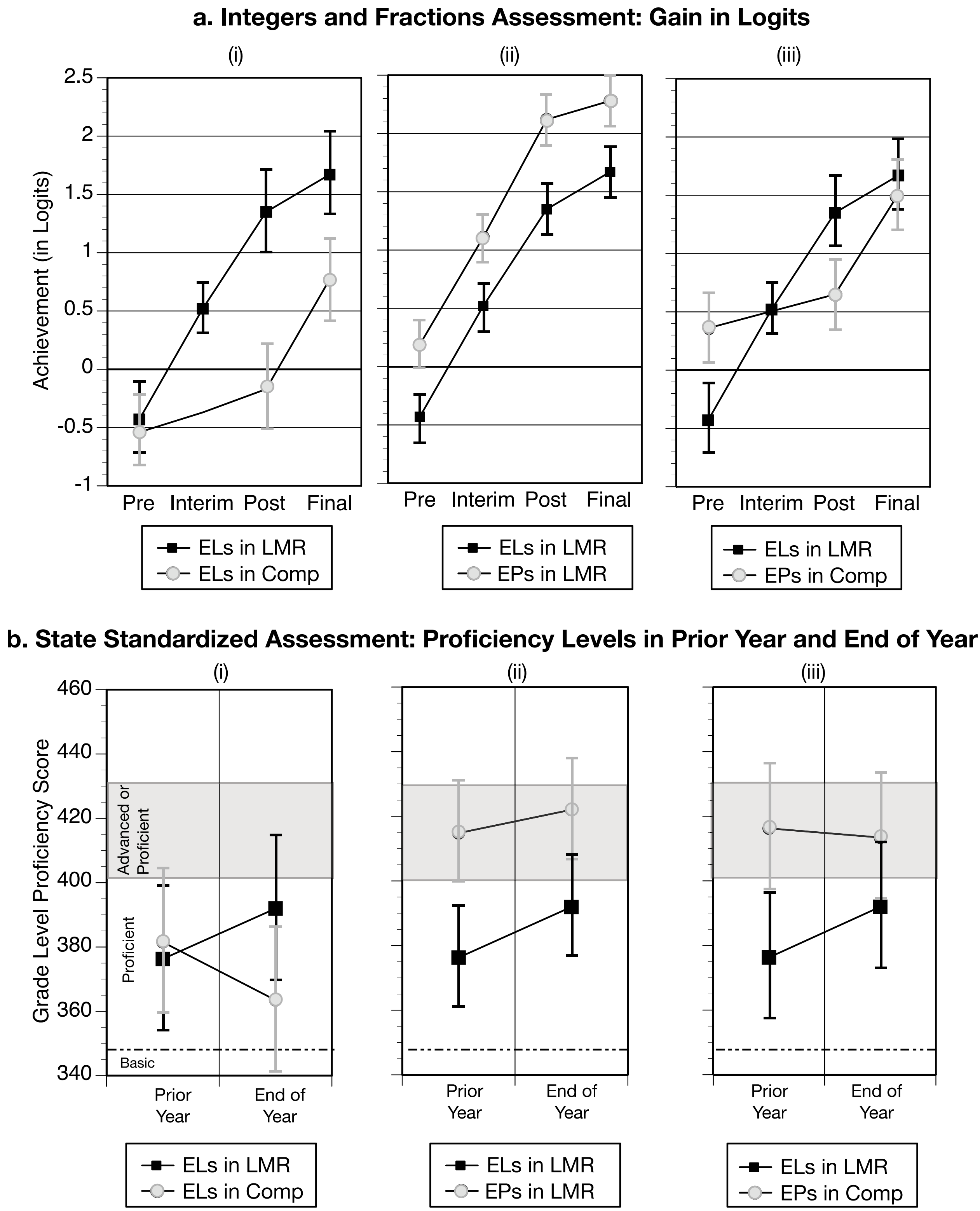

FIGURE 7. *A comparison of student groups using (a) the integers and fractions assessment (modeled achievement in logits over four assessments) and (b) the standardized assessment (modeled grade level proficiency over two assessments). The panel sequence compares (i) English learners (ELs) in* Learning Mathematics Through Representations *(LMR) classrooms with ELs in comparison (Comp) classrooms, (ii) ELs in LMR classrooms with English proficient students (EPs) in LMR classrooms, and (iii) ELs in LMR classrooms with EPs in Comp classrooms.*
Note. A flat slope in each of the panel b figures would reflect an expected one-year gain in grade-level mathematics proficiency.

Table 1
Summary of Principal Findings Keyed to Expectation and Measure

Expectation	Measure	Principal Finding
(1) LMR efficacy for ELs. ELs who participate in LMR will show greater gains in mathematics than ELs in comparison classrooms.	Integers and fractions	Expectation corroborated (Figure 7a.i).
	Standardized state assessment	Expectation corroborated (Figure 7b.i). ELs in LMR gained in proficiency, whereas ELs in comparison classrooms declined in proficiency.
(2) LMR efficacy for ELs versus EPs. ELs and EPs who participate in LMR will show similar gains in mathematics.	Integers and fractions	Expectation corroborated (Figure 7a.ii). ELs and EPs each showed strong learning gains with no detectable differences in rates of learning.
	Standardized state assessment	Expectation corroborated (Figure 7b.ii). ELs and EPs showed no detectable difference in rates of learning as a function of language status.
(3) LMR efficacy in reducing the achievement gap between ELs in LMR classrooms and EPs in comparison classrooms. The post-intervention achievement gap between ELs who participate in LMR classrooms and EPs who participate in comparison classrooms will be reduced relative to their pre-intervention gap.	Integers and fractions	Expectation corroborated (Figure 7a. iii). ELs in LMR grew more than EPs in the comparison group. The growth difference was large: The pretest EL-EP gap was reversed at posttest, with no difference between the EL (LMR) and EP (comparison) performances at final test.
	Standardized state assessment	Expectation partially corroborated (Figure 7b.iii). Achievement gap narrowed: There was no statistical difference between EL (LMR) and EP (comparison) at end of year, though if the statistical test had greater power, we may have documented that the comparison EPs maintained greater achievement than the LMR ELs. Limitation of the analysis: The statistical test appears underpowered.
(4) LMR efficacy in supporting gains on items that included and did not include number line representations. ELs (and EPs) who participate in LMR will develop integers and fractions knowledge not solely linked to the number line.	Integers and fractions assessment	Expectation corroborated (Figure 8). ELs and EPs in LMR classrooms showed strong learning gains on items that contained number lines and those that did not.

Note. LMR = *Learning Mathematics Through Representations*; ELs = English language learners; EPs = English proficient peers.

Integers and fractions assessment. Figure 7a.ii contains a graph of student achievement on the integers and fractions assessment for ELs and EPs who participated in LMR classrooms. The figure reveals similarity in the growth trajectories for ELs and EPs. To further analyze whether gains in achievement were similar for the LMR language proficiency groups, we examined differential gains over the year for ELs and EPs. We found, consistent with our expectation, strong effects for treatment from pre to post for both ELs ($ES = 0.86, p < .001$) and EPs ($ES = 1.00, p < .001$) and pre to final for ELs ($ES = 0.48, p < .001$) and EPs ($ES = 0.57, p < .001$). Relatedly, the interaction between language status and treatment was not statistically significant from pre to post ($p = .277$) or pre to final ($p = .489$). We note that the analysis of interaction effects was somewhat underpowered and although our analysis may not have revealed interaction effects if they do exist, the rates of change for ELs and EPs were similar.

Standardized assessment. To examine whether LMR is equally effective for supporting grade-level mathematics achievement for ELs and EPs, we examined the differential gains for ELs and EPs in LMR on the standardized assessment. Figure 7.b.ii shows that ELs in LMR demonstrated slightly greater growth than EPs in LMR during the year—that is, a slightly sharper slope. However, a statistical contrast between the slopes was not significant ($p = .443$), meaning that our evidence does not reveal that ELs in LMR would "catch up" to EPs in LMR. Nonetheless, the rate of learning for ELs was similar to EPs, which suggests that their learning opportunities were functionally similar, considering their initial achievement levels. These results corroborate the findings produced with the integers and fractions assessment: LMR supported similar gains for ELs and EPs.

In a follow-up analysis, we examined the possibility that a ceiling effect contributed to the finding of no differences in rates of learning for ELs and EPs in LMR classrooms. EP students' scores were skewed towards the maximum, which leaves less room to detect growth compared to ELs and poses a threat to validity of our analysis. To address the threat, our follow-up replicated our original analysis but excluded all students who scored in the top quartile of the prior year assessment. The follow-up corroborated the validity of our original analysis: ELs in LMR again demonstrated slightly greater growth than EPs in LMR, but again this difference was not statistically significant ($p = .135$).

(3) Achievement Gap Between ELs in LMR Classrooms and EPs in Comparison Classrooms

To investigate the potential role of LMR in attenuating the EL-EP achievement gap (our third expectation), we contrasted the achievement gains of LMR ELs with the achievement gains of comparison EPs. We reasoned that EP students in comparison classrooms could be treated as a benchmark for EP achievement in high quality standard curricula. Contrasting EL LMRs with EP Comparisons (that were engaged with a widely used, well-regarded curriculum) would allow us to assess the practical significance of LMR—whether LMR might attenuate the persistent achievement gap. This contrast would not be relevant if LMR were implemented at scale, whether within all classrooms in a district, state, or nation (of course, we regard such universal adoption an unlikely scenario). As we report next, the findings indicate that the size of the LMR treatment effects helped ELs in LMR classrooms catch up to their EP peers in comparison classrooms.

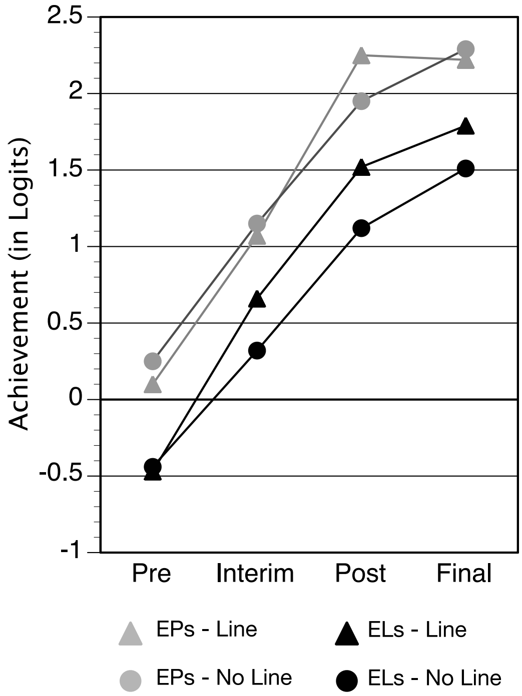

FIGURE 8. *Comparison of ELs' and EPs' performances on items containing and not containing number line representations who participated in LMR classrooms.*
Note. ELs = English language learners; EPs = English proficient peers.

Integers and fractions assessment. Figure 7a.iii is a graph of the estimated mean scores (in logits) on the integers and fractions assessment for LMR ELs and comparison EP students. At pretest, there was a statistically significant EL-EP achievement gap estimated at 0.76 logits[3] (SE = 0.14, $p < .001$), which corresponds to an achievement gap of 0.46 SD. In the first part of the academic year, the achievement of ELs in LMR grew sharply relative to the achievement of EP students in the comparison group. By the post assessment, the EL-EP achievement gap reversed, with ELs in LMR showing 0.70 logits (SE = 0.32) greater estimated achievement than EP students in the comparison group (p = .027, ES = 0.43). The reverse achievement gap narrowed between posttest and final test, and at the end of the school year, the final test showed no statistical difference between ELs in LMR and EP students in the comparison group (p = .608). In summary, ELs in LMR grew an estimated 0.95 logits (SE = 0.16) more over the year than EPs in the comparison group ($p < .001$, ES = 0.58), and the findings indicate that the EL-EP achievement gap in integers and fractions achievement was effectively eliminated.

Standardized assessment. Figure 7b.iii shows the change in estimated achievement gaps from prior end-of-year to end-of-year standardized assessments, revealing a reduction in the gap over the year of LMR implementation. In the prior year, there was a statistically significant EL-EP achievement gap estimated at 35.6 points (SE = 10.6, $p < .001$), which corresponds to an achievement gap of 0.39 SD in grade-level proficiency. The gains for LMR ELs from the prior year to the end-of-year test, represented by the positive slope, resulted in a narrowing of the achievement gap. On the end-of-year test, the estimated achievement gap between LMR ELs and comparison EPs was only 21.5 points (SE = 19.4), an achievement gap of 0.24 SD which was not statistically significant (p = .268). Although the results suggest that participation in LMR was associated with a 38.5% reduction in the achievement gap, a finding consistent with the prior analysis using the integers and fractions assessment, we interpret the results with caution. Null findings—the larger p value for the end of year comparison—should not be taken as evidence that no difference exists between ELs and EPs general mathematics achievement (i.e., that LMR completely eliminated the achievement gap).

(4) ELs in LMR Classrooms Gains on Items That Included and Did Not Include Number Line Representations

To investigate our fourth expectation—that ELs in LMR classrooms would show strong gains in integers and fractions regardless of item format (whether or not items contain number lines), we analyzed EL students' performance on two item-type subscales. Figure 8 contains mean estimated scores of EL and EP students in LMR classrooms on number line and no number line items across pre, interim, post, and final assessments. The figure shows strong gains for ELs and EPs on both item types over the course of the school year. However, ELs performed better on number line items than non-number line items on the interim, post, and final assessments. For EL students, the difference between line items and no line items was not statistically significant at pretest (p = .807), but ELs scored higher on the set of line items than the no line items on the interim (p = .015, ES = 0.21), post (p = .003, ES = 0.24), and final (p = .042, ES = 0.17) assessments. Perhaps the stronger performance on number line items for ELs is to be expected given the support that the graphical representation of the line would have come to provide for EL students on assessment items, given their differential mastery of English combined with support they received with the LMR curriculum.

Discussion

In this article, we reported new findings regarding the efficacy of LMR for English language learners (ELs), a subgroup of the students who participated in the LMR efficacy study (Saxe, Diakow, et al., 2013). The collective findings from two assessments revealed that EL students profited more in LMR classrooms than in the comparison classrooms (that featured a well-regarded curriculum in widespread use), that the growth in achievement of EL students kept pace with the EP students in the same LMR classrooms, and that the EL-EP achievement gap was eliminated (integers and fractions assessment) or reduced (standardized assessment) when the achievement of EL students in LMR classrooms was compared to that of EP students in the comparison classrooms. Moreover, when we analyzed student gains on the integers and fractions measure for items that included and did not include number line representations, we found that EL and EP students in LMR classrooms made strong gains on each item type, even though LMR classrooms used number lines as the

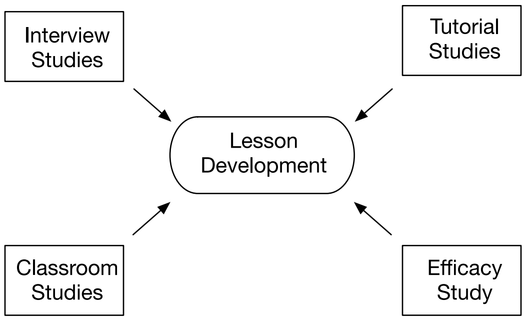

FIGURE 9. *The model for design-based research that supported the development of the* Learning Mathematics Through Representations *curriculum unit: Interview, tutorial, classroom, and efficacy studies.*

principal representational context. Although learning gains on both item types were strong for EL students in LMR classrooms, we did note that EL students achieved greater scores on the number line items at interim, post, and final tests, perhaps reflecting the support that number lines provided for these students after the LMR intervention.

How did LMR support the mathematics learning gains of EL students? We argue that features of LMR differ in important ways from the features of many published curriculum materials, enabling ELs to use their partial mastery of the English language to engage and build upon their mathematical intuitions. Over the 19-lesson sequence, LMR engages students with the coordinated use of visual representations (number lines), verbal and written representations (mathematical definitions with reference to the number line), and sensorimotor representations (movement/manipulation of linear representations [e.g., Cuisenaire rods] on the number line). Further, the five-phase lessons afford teachers opportunities to assess and integrate student reasoning in discussions and to adapt their instruction as students with diverse understandings and linguistic proficiencies reason publicly with varied representational formats. Thus, the lesson sequence increases the likelihood that all students, including ELs who may have difficulty accessing traditional mathematics curricula, will have multiple opportunities over time to engage with complex mathematical ideas and build proficiency.

Why did EL students in LMR show marked gains on a general assessment of math proficiency (the state standardized assessment) when the intervention was focused on integers and fractions? One possibility is that LMR teachers developed more inclusive instructional practices in the fall, and then sustained their use of LMR design principles through the remainder of the year. Another possibility is that the gains that EL students made in the fall seeded a developmental process that enhanced some ELs' ability to engage with mathematics in other domains and with other materials. We expect that both factors may have contributed to EL student achievement.

Similar to other design-based research projects, LMR was created to elicit important learning phenomena in order to investigate theoretically-motivated instructional strategies (cf. Abrahamson, 2009; Abrahamson & Lindgren, 2014; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Gutiérrez & Jurow, 2016; Hall & Jurow, 2015; Kelly, 2006). Importantly, LMR's orchestrated design approach has some unique features that supported student learning opportunities. The LMR project began with grounded conjectures about the value of the number line for a strong instructional treatment of integers and fractions (Saxe, de Kirby, Le, et al., 2015). Early classroom studies of pilot lessons featuring number line representations revealed the diversity of students' reasoning as students grappled with complex mathematical ideas (e.g., Saxe et al., 2007; Saxe et al., 2009). Subsequent interview studies yielded systematic evidence of the patterns of student reasoning about integers and fractions in different representational contexts (e.g., Saxe, Shaughnessy, Gearhart, & Haldar, 2013). Tutorial studies enabled the design and validation of productive learning trajectories when students are provided with visual, definitional, and embodied representational supports (Saxe et al., 2010). Results from this collection of studies then led us back to the classroom to partner with teachers in developing lesson sequences (Saxe, de Kirby, Le, et al., 2015). When the curriculum was complete, we conducted an efficacy study that provided quantitative evidence of LMR's effectiveness (Saxe, Diakow, et al., 2013) and qualitative analyses of effective classroom practices as LMR teachers engaged their students with lessons (Saxe, de Kirby, Kang, et al., 2015). Our orchestrated model of interview, tutorial, classroom, and efficacy studies in our design-based research (see Figure 9) has proven to be a powerful methodology for developing educational innovations. The present study of ELs in LMR and comparison classrooms provides further support for the utility of the research-based design approach.

Our current study on ELs cannot identify specific features of LMR that supported ELs' gains in achievement. We treated LMR and comparison groups as "packaged variables" (Cole, 1990; Whiting, 1976) since it was not possible to isolate visual representations, verbal definitions, participation structures, amount of time spent with LMR teachers, or other distinctive features of LMR. Thus, we could not determine whether any one of these features alone or interaction with one another enhanced learning opportunities for ELs, or whether the interplay between these variables varied over EL students. What we can assert is that the emergent environments in LMR classrooms engaged ELs in ways that provided more equitable learning opportunities. We regard our findings, methods, and design-based research approach as important resources for researchers and professionals developing instructional approaches that engage all children with rich learning opportunities.

NOTES

The authors contributed equally to this article. The research reported here was supported in part by the Institute of Education Sciences grant R305B070299 and R305B090026 to University of California, Berkeley (PI: Geoffrey B. Saxe). The opinions expressed are those of the authors and do not represent views of the Institute of Education Sciences or the U.S. Department of Education. We are grateful to the participating teachers and students and to the many contributors to the project. Rick Kleine and Jenn Pfotenhauer served as collaborating teachers in the development of the lesson sequence. Professor Maryl Gearhart and a team of former doctoral students at University of California, Berkeley contributed both to the research and to curriculum development. The former doctoral students who participated in lesson development include Ronli Diakow, Darrell Earnest, Lina Chopra Haldar, Bona Kang, Katherine Lewis, and Yasmin Sitabkhan. Additional former graduate student contributors include Nicole Leveille Buchanan, Anna Casey, Jennifer Collett, Kenton de Kirby, David Torres Irribarra, Marie Le, Amanda McKerracher, Meghan Shaughnessy, Alison Miller Singley, and Ying Zheng. We are grateful to project consultants, Professors Deborah Loewenberg Ball and Hyman Bass, for their encouragement and feedback during formative stages of our design-based research. We also extend our appreciation to the article reviewers whose feedback led to improvements on earlier versions of this article.

[1]In our sample, the prior-year and end-of-year assessments showed similar means and standard deviations allowing us to treat the measurement scales as comparable (See online supplement S7 [p. S7-2, available on the journal website] for additional information.).

[2]See online supplement S12 (available on the journal website) for standardized treatment effects across all waves and assessments.

[3]Pretest achievement gaps (for both the integers and fractions assessment and the standardized assessment) were estimated from the final models each fit without a dummy variable for assignment to treatment.

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Manuscript received May 22, 2018
Revisions received December 4, 2018; May 9, 2019; June 28, 2019
Accepted July 11, 2019