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Representing Fractions with Standard Notation: A Developmental Analysis

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This study had two purposes: (a) to investigate the developmental relationship between students' uses of fractions notation and their understandings of part-whole relations; and (b) to produce an analysis of the role of fractions instruction in students' use of notation to represent parts of an area. Elementary students ($n = 384$) in 19 classes participated in the study. Pre- and posttests were administered before and after fractions instruction, and key lessons were recorded with videotape and field notes. Students' written responses were coded in two ways: for the forms of the notations (e.g., use of numerator, denominator, and separation line), and for the concepts captured by the notations (e.g., part-whole, part-part, or other kinds of relations). The lessons captured on videotapes and in field notes were rated with respect to their alignment with principles supported by reform frameworks in mathematics education (e.g., opportunity to build understanding of fractions concepts, ongoing assessment of student understanding). Our analyses indicated (a) notation and reference were acquired somewhat independently, and (b) classroom practices that built on students' thinking were more likely to support shifts toward normative uses of notation.

Key words: Elementary, K–8; Fractions; Instructional intervention; Professional development; Rational number/representations; Reform in mathematics education; Teaching effectiveness

Mastering fractions is a major hurdle for students in the middle elementary grades and beyond (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980; Moss & Case, 1999), and one contributing factor may be students' difficulty acquiring flexible use and understanding of written notations for fractions (Hiebert, 1988). Although students' problems with written notation for fractions are widely acknowledged (Hiebert, 1989; Kouba, et al., 1988), we have few systematic studies of developmental patterns in their use of notation or the role of instruction in their development of notation. Those studies that do exist, although providing considerable insight, tend to be limited to case studies (e.g., Ball, 1993; Mack, 1990, 1995).

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In this study of elementary students in 19 classrooms, we investigated the development of students' use of standard notation to represent fractional parts of areas. We report a longitudinal study of relations between fractions instruction and students' developmental trajectories of change from before instruction to after instruction. To organize our inquiry, we introduce a framework that distinguishes between two aspects of students' written representations of fractional parts of areas—notation and reference.

Notation consists of marks or symbols (e.g., Hindu-Arabic numerals) as well as rules for concatenating these marks. Notations for fractions, which is our focus here, vary across cultural groups and historical time periods. The notational system used in contemporary Western societies has its provenance in ancient India, and other cultures have developed different notational forms that make use of different characters and rules of concatenation. Each system—its elements and their arrangements—has particular conventions at any given time, and each has the potential to be altered and reorganized over the course of social history (see Menninger, 1969; Smith, 1925).

By *reference* we mean the conceptual work of using a notational form to point to or index physical objects or mathematical ideas. When individuals use Hindu-Arabic notations to refer to whole numbers, they connect these notations to a system of values in which additive relations are of primary importance, as manifested in the addition or subtraction of natural numbers (e.g., 1 bead + 2 beads = 3 beads). When individuals use Hindu-Arabic notations to refer to fractions, they connect these numerals to a system in which multiplicative relations are of primary importance. For example, equivalence is defined in terms of multiplicative relations between parts and wholes such that 1/2 = 2/4 = 3/6, etc., and the numerators and denominators are only elements in the expression of a quotient or multiplicative relation between values. Multiplicative relations are constitutive of the conceptual field of rational number, including fractions and a family of related rational number constructs such as part-whole relations, ratios, quotients, measures, and operations (Ball, 1993; Behr, Harel, Post, & Lesh, 1992; Behr, Lesh, Post, & Silver, 1983; Confrey & Smith, 1995). Such multiplicative relations are fundamental to a treatment of fractions for discrete quantities, like a multiplicative relation between elements of discrete sets (e.g., two thirds of a populations of individuals) or for continuous quantities, like multiplicative relations between portions of continuous units, (e.g., two thirds of an area).

USING STANDARD NOTATION FOR FRACTIONAL PARTS OF AREA

The focus of our article is the development of students' notations for fractional parts of area. The partitioning of an area into fractional parts is a common vehicle for elementary fractions lessons (e.g., Corwin, Russell, & Tierney, 1991) and a productive arena for research on relations between notation and reference.

A familiar task that employs a fractions model is often referred to as *fair sharing* (e.g., Chen & Okamoto, 1998; Empson, 1999; Hunting, Davis, & Pearne, 1996;

Pepper & Hunting, 1998; Piaget, Inhelder, & Szeminska, 1960; Pothier & Sawada, 1983; Saenz-Ludlow, 1994; Streefland, 1991; Tzur, 1999). Fair sharing is a common practice outside of school as well as in the classroom, and, prior to formal instruction in fractions, students already show some knowledge about fractional parts of area from their experiences sharing foods that have circular (e.g., pizza) or rectangular (e.g., cakes) shapes. When solving fair-sharing tasks, young children show some incipient understanding of partitioning area into equal parts, especially when partitioning a quantity into halves (Pitkethly & Hunting, 1996), although they have difficulties with more complex distributions (Piaget, et al., 1960). Young children also show emerging understandings of the inverse relation between number of people and size of share. For example, in a series of studies, Sophian, Garyantes, and Chang (1997) demonstrated that 5-year-olds could be successfully taught that the greater number of people, the smaller the shares in fair-share problems. Further, Mix, Levine, and Huttenlocher (1999) revealed that 5-year-olds show competence with the addition and subtraction of fractional parts of areas. Mix and colleagues presented children with a fractional amount (one fourth, two fourths, or three fourths of a circle) and then hid it behind a screen; next, they visibly added to or subtracted from the screened quantity. Five-year-olds selected the correct amount from a multiple-choice display at levels that were better than chance.

Researchers have used longitudinal, qualitative studies to investigate developmental change in students' understandings of fractions with area models and the way instruction may support change. Some of these were teaching experiments in which a researcher worked closely with students in tutorial interactions over many sessions (Olive, 1999; Saenz-Ludlow, 1994, 1995; Tzur, 1999), whereas others were classroom-based studies that tracked the developing understandings of focal students over the course of multiple lessons (Ball, 1993; Empson, 1999, 2003). These studies indicate that students use their knowledge of whole numbers (i.e., additive relations) and other informal knowledge to interpret fractions problems and face many challenges in their efforts to construct understandings of multiplicative relations and represent fractional parts of both discrete and continuous quantities with new forms of notation.

Mack's 1995 study provides an illustration of a teaching experiment that focuses on written notation. Mack engaged 7 third and fourth graders in six individualized lessons over the course of a 3-week period, helping students connect their informal knowledge with written notation by posing problems and asking questions. From her clinical interviews, Mack reported that these students faced two challenges coordinating written notation with fractions concepts. On the one hand, the students tended to interpret written notations in whole number terms. For example, one child remarked that the notation for five eighths of an area could be written as either "5" or "5/8," explaining that "it doesn't matter. It's the same thing" (Mack, 1995, p. 435). The child's focus may have been on the 5 parts as discrete "pieces," without concern to represent the whole in the notation or concerns for whether their representation should accommodate different sizes of parts. On the other hand, students tended to interpret whole number notation of mixed number expressions in terms of a language of fractions. Thus, they consistently interpreted the “2” in the subtraction expression “2 - 3/8” as two eighths. Through her methods of posing problems and asking questions that built on student thinking, Mack was quite successful supporting students’ differentiation and coordination of fractional quantities and conventional notation for fractions.

As an example of classroom-based research, Ball’s study (1993) captures students’ interpretations of fraction notation in the context of instruction. From records of her work as a third-grade teacher, Ball reported that her students initially made sense of notations for fractions in ways that were at odds with normative approaches. Many of these notational patterns indicated that students had not yet differentiated continuous and discontinuous quantity. For example, one student interpreted 3/4 as operations on discrete sets—to make four groups of three and then take away all but one group. Other students assumed that 1/4 could only apply to one stereotypic shape, a quarter circle, an assumption that circumvented an analysis of parts and wholes. Ball’s instruction was designed to support students’ emerging understandings of relations between fractions representations and fraction concepts. Her method was to pose problems using various models (e.g., area models, discrete models), and guide inquiry through questions about these representational contexts. Ball reports evidence that students shifted from an inexplicit and tacit intuition of ideas like 1/4 as a particular shape to more principled understanding of fractions notations as expressions of part-whole relations. Yet students were challenged when coordinating new notations with changing conceptions of part-whole relations, as Ball illustrates with an anecdote about one student who interpreted the meaning of “halves” as “twoths.” This child’s insight reflects an overgeneralization of linguistic rules for constructing fraction words along with a growing appreciation of the mathematical basis for fractions notation.

Although Mack (1995) and Ball (1993) reported some success in students’ understanding, their studies demonstrate that students are challenged in their efforts to learn fractions. Some researchers have argued that the emphasis on notation for countable whole numbers in the primary grades interferes with students’ re-conceptualization of notation to serve part-whole functions in the upper-elementary grades (Mix et al., 1999; Saenz-Ludlow, 1994; Sophian et al., 1997). When the focus shifts to fractions, students are expected to create new kinds of expressions with numerals in fractions notation and to use these numerals for new referential functions—the representation of part-whole relations both for discrete quantities like beads and continuous quantities like area. Instruction that fails to engage students in investigating the connections between new forms of notation and new part-whole quantitative referents may contribute to persisting difficulties with fractions (Hiebert, 1988).

PURPOSE OF THE STUDY

Our study of 384 upper-elementary students extends the findings from prior research on fractions notation and reference in two ways. First, we examined developmental relations between students' use of fractions notation and students' understandings of part-whole relations. We were interested in determining whether these two developmental trajectories might be somewhat independent. For example, some students who have developed an understanding of part-whole relations might produce a notation for a shaded fourth of a square divided into four equal parts as 1/4, whereas others might label it 4/1 or 1-4; the latter notations are unconventional but nevertheless convey a concern for part-whole relations. Conversely, some students who have developed the conventional notations form might use it to refer to discrete countable pieces. For example, they might label the same shaded fourth as 1/3 to refer to 1 shaded piece and 3 unshaded pieces. Second, using a pre-posttest design, we investigate the role of fractions instruction in students' notations for parts of area and the relations that students create between notations and referents. In relation to prior qualitative studies like those of Mack (1995) and Ball (1993), these latter data contribute a more systematic analysis of relations between instruction and trajectories of change in notation. These data also contribute to a growing literature that examines relations between instructional approach and student achievement, by comparing the growth of students in inquiry classrooms vs. more traditional classrooms that emphasize skills (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Hiebert & Wearne, 1993; Riordan & Noyce, 2001; Saxe, Gearhart, & Nasir, 2001; Saxe, Gearhart, & Seltzer, 1999). These studies provide support for the inquiry methods, but none has produced an analysis of trajectories of change with a focus on notation for fractions.

Our focus was on students' representations of fractional parts of continuous quantities. To investigate relations between notations and reference in this context, we used two contrasting types of area representations and asked students to produce written notations for each type. Examples of each representation type are in Figure 1. *Equal Area problems* consist of shapes partitioned into equal-sized parts. *Unequal Area problems* have parts of unequal sizes, and are a type of problem some have described as having distracting or incomplete cues (Behr et al, 1983; Kieren, Nelson, & Smith, 1985); their use affords an analysis of students' interpretation of size relations between parts of areas. Students who are interpreting areas in whole number terms might respond correctly to an Equal Area problem simply by counting parts. The responses "1/4" or "1-4" or "4/1" are ambiguous evidence of reference, and it is in students' response to the Unequal Area problems that we determine

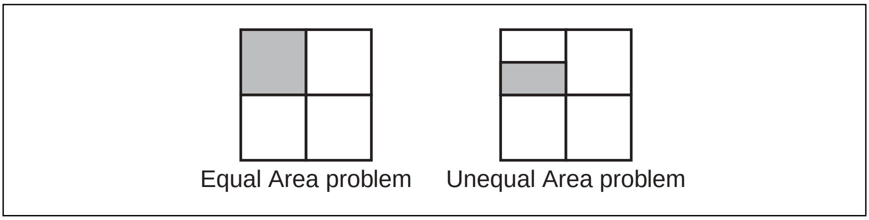


Figure 1. Examples of Equal and Unequal Area problems.

whether they differentiate parts of area as discrete or continuous quantities in their representations (e.g., 1/5 or 1/8). We examined students' reference to parts and part-whole relations for each problem type, as well as their use of conventional (e.g., 1/4) vs. unconventional (e.g., 1-4 or 4/1) notation.

DESCRIPTION OF THE STUDY

Our study addressed three questions related to relations between notational forms and reference. First, when students produce part-whole reference on equal and unequal area problems, are they treating parts of area as discrete or continuous quantities? Second, do conventional notation and part-whole reference develop in concert, does one precede the other, or is there some independence? Further, does progress in either notational or referential aspects of written notation affect progress in acquiring normative use of the other? For example, are students who begin instruction with advanced reference more likely to show greater gains in notation? Third, for students who begin fractions instruction with similar notational approaches, how are their trajectories affected by participation in classrooms that stress inquiry as contrasted with classrooms that stress skills?

To address these questions, we followed students from 19 classrooms in Grades 4, 5, and 6, administering equal and unequal area problems prior to and at the conclusion of their fractions units. The classrooms varied with regard to the mathematics curriculum used: about two thirds of the classrooms worked with an inquiry unit, *Seeing Fractions* (Corwin et al., 1991), that supported investigations of the quantitative meanings of standard notation in the context of area and linear models, and about one third used traditional textbooks oriented to fractions skills and definitions. The classrooms varied in the extent to which they supported inquiry into fractions concepts and notations. Coding schemes for observations of whole class fractions lessons provided indicators of the depth of conceptual content in the lessons and the extent to which teachers elicited and integrated students' mathematical thinking in class discussions (Gearhart et al., 1999). We used these indicators to create three contrasting sets of classrooms. In both the High Inquiry and Low Inquiry classrooms, students worked with *Seeing Fractions*, but students were engaged in more conceptually rich discussions in High Inquiry classrooms. In Traditional classrooms, students used textbooks, and were less likely to be engaged in conceptually rich discussions; emphasis was more on acquiring procedural skills.

METHODS

Participants

Our study draws on a corpus of data collected in a research project entitled the Integrated Mathematics Assessment Study (Gearhart et al., 1999; Saxe et al., 1999; Saxe et al, 2001). The study followed students in 19 upper-elementary classrooms during curriculum units on fractions. The sample contained 384 students of mixed-language backgrounds and ethnicities whom we followed from a pre- to a postunit assessment. In the entire sample, 64% were Latino, 14% were White, 8% were African American, and 7% were Asian. The median grade level of the upper-elementary classrooms was Grade 5.

The classrooms used one of two types of curriculum materials. Fifteen classrooms (involving 315 students) used curriculum units aligned with the state of California's 1992 Mathematics Framework (California Department of Education, 1992), *Seeing Fractions* (Corwin, et al., 1991) and *My Travels with Gulliver* (Kleiman & Bjork, 1991). The teachers in these classrooms had prior experience teaching these units and had agreed to teach the units again during the project year and to implement teaching and assessment methods discussed in staff development meetings. Four additional classrooms (involving 69 students) used traditional texts in which skills were emphasized in chapters on fractions, measurement, and scale. These teachers had no experience with reform curriculum, were committed to textbook instruction, and had agreed to use textbooks again during the project year. Across groups, we attempted to balance classrooms for grade level and language proficiency of the students.

For the study reported here, we partitioned classrooms into three categories of classroom practice: (a) use of inquiry curriculum in ways that were more aligned with inquiry principles (High Inquiry, 8 classrooms); (b) use of inquiry curriculum in ways that were less aligned with inquiry principles (Low Inquiry, 7 classrooms); (c) use of traditional curriculum in ways that were less aligned with inquiry principles (Traditional, 4 classrooms). Judgments of alignment with inquiry principles were derived from our ratings of whole class lessons. Based on two sets of field notes (introductory area model and linear model lessons) and one videotaped lesson (on addition of fractions), we analyzed whole-class lessons using coding schemes developed by Gearhart et al. (1999). The schemes allowed us to capture two aspects of classroom practice valued by mathematics education reform (National Council of Teachers of Mathematics [NCTM], 1989, 2000): (a) the depth of conceptual content (e.g., part-whole relations) in treatments of problem-solving, and (b) the extent to which discussions built on student understandings.

Assessing Students' Representations

Data were drawn from a more comprehensive paper-and-pencil, pre- and postassessment of fraction knowledge reported in Saxe et al. (2001). The relevant data consisted of students' written notations for two types of problems. Four problems used geometrical shapes partitioned into equivalent areas (Equal Area problems), depicted in Figure 2 (a–d), and the remaining three problems used geometrical shapes that were partitioned into sections in which the areas were not equivalent (Unequal Area problems), depicted in Figure 2 (e–g). For each problem type, the child was presented with the prompt, "For each picture below, write a fraction to show what part is gray." The data used for these analyses were collected just prior to initiation of instruction and just after the completion of instruction. Teachers

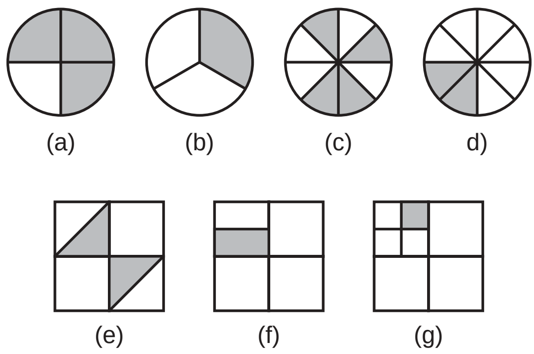


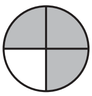
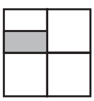
Figure 2. Problems used in the study: Equal Area (a)-(d) and Unequal Area (e)-(g).

varied in how they managed their time, some drawing instruction over a longer period than others. The periods varied from 6 to 12 weeks.

Table 1 contains brief definitions and examples used in our coding scheme for the fraction problems in Figure 2. Our coding schemes for *reference* differed slightly for Equal Area and Unequal Area problems. For Equal Area, each answer received one of five codes. *Part-Whole* (P-W) reference were answers in which one of the two digits matched the number of parts that were gray and the other digit matched the total number of parts or its fraction equivalent; for example, the answer for the Equal Area problem in Table 1 may be written in ways such as 3-4; 4-3; 3,4; 3/4; or even 6/8. *Part-Part* (P-P) reference were answers in which one of the two digits matched the number of gray parts, and the other digit matched the number of white parts (e.g., the three-fourths shaded area in the table may be written as 3-1; 1-3; 3 1; or 1/3). *Integer* (I) reference were answers in which a single digit matched either the number of gray or white parts (e.g., 3 shaded areas may be written 3). *Don't Know* indicated that they did not know the answer, which they typically indicated by writing "dk" on their answer sheets. *Other* was used for answers that did not conform to the other categories; these included answers in which the digits did not match the number of gray or white parts in the targeted figure or for which no numerical notation was produced.

For Unequal Area problems, the Part-Whole category was separated into two types, *Part-Whole: Continuous* (P-W:C) and *Part-Whole: Discrete* (P-W:D). Part-Whole: Continuous reference accommodated the unequal areas by treating area as a continuous quantity in a written representation. Thus, the representation of the gray area in the Unequal Area problem in Table 1 would be 1/8 or its fraction equivalent. In contrast, Part-Whole: Discrete reference treated the grayed area as a discrete quantity, treating parts as equivalent regardless of size. Thus, the representation of the gray area in the Unequal Area problem in Table 1 would be 1/5 or its equivalent, since one of five parts is shaded.

Table 1
Reference and Notation Codes for Equal Area and Unequal Area Problems

			Notation codes and sample representations	
Example of problem type	Reference code	Definition of reference code	Unconven-tional	Conven-tional
Equal Area 	Part-Whole (P-W)	Notation captures the number of gray parts and number of all parts in whole	3-4; 4-3; 3,4; 3 4; 6,8	3/4; 6/8
	Part-Part (P-P)	Notation captures number of gray parts and number of white parts	3-1; 1-3; 3 1; 1 3	3/1
	Integer (I)	Notation captures the number of gray parts	3; 1	—
	Don't Know (dk)	"Don't Know" written as a response	—	—
	Other	None of above	7	7/9
Unequal Area 	Part-Whole: Continuous (P-W:C)	Notation captures unequal size relations in the repre-sentation of part whole relation	1-8; 8-1; 8 1; 8,1 16,2; 2-16	1/8; 2/16
	Part-Whole: Discrete (P-W:D)	Notation captures the number of gray parts and number of all parts in whole (without accommodating different sizes of parts)	1-5; 5-1; 5 1; 1,5	1/5
	Part-Part (P-P)	Notation captures number of gray parts and number of white parts	1-4; 4 1; 4-1	1/4
	Integer (I)	Notation captures the number of gray parts	1; 4	—
	Don't Know (dk)	"Don't Know" written as a response	—	—
	Other	None of above	7	7/9

Our coding schemes for *notation* were identical for Equal and Unequal Area problems. For each problem, students' notations received one of two codes. *Conventional* was assigned to notations that contained a numerator and a denominator separated by a horizontal line (—) or a slash (/). *Unconventional* was assigned to notations that did not meet the criteria for conventional such as single digits, and numbers separated by a hyphen, a colon, or any mark other than a horizontal line or slash. Examples of these kinds of responses are presented in Table 1.

Following the coding, we assigned students to categories based on the consistency of their codes across tasks. For Equal Area problems, if students received the same reference code on at least 3 of the 4 problems (which we considered to be a "modal" response), they were assigned that code as their overall code; all other students were assigned to the category of No Modal. We used the same procedure for notation codes on Unequal Area problems. For the Unequal Area problems, we used parallel procedures, but the criterion for modal categorization was consistency on 2 of the 3 problems.

RESULTS

The results are organized in three sections. We focus first on patterns of part-whole reference across the Equal Area and Unequal Area problems to determine if performance on Unequal Area captures students' differentiation between fractional parts of discrete and continuous quantities. We then consider evidence for the developmental independence of notation and reference. Third, we examine relations between instruction and changes in students' notation and reference.

Patterns of Part-Whole Reference

How do students who create part-whole reference on Equal Area problems interpret the unequal-sized parts on Unequal Area problems? To answer this question, we cross-tabulated students' modal response categories on these two problem types, collapsing some of the coding categories from Table 1 to focus just on students' reference to part-whole relations. For instance, for the Equal Area problems the categories Part-Part, Integer, Don't Know, and Other were collapsed to form the Not Part-Whole category. Table 2 contains the results for the pretest data using percentages of students.

The findings in Table 2 reveal that, of the students who referred to part-whole relations on Equal Area problems, many did not on Unequal Area problems—a pattern indicating that many students were not differentiating between discrete and continuous quantities. The bottom row of the table shows that the students who produced Part-Whole responses on Equal Area problems produced a range of responses on the Unequal Area problems. Only 9% of students interpreted both

Table 2
Percentage Distribution of Students' Modal Reference Categories Collapsed Across Equal and Unequal Area Problems Based on Pretest Data

	Unequal Area problems		
Equal Area problems	Not Part-Whole	Part-Whole: Discrete	Part-Whole: Continuous
Not Part-Whole	30	1	0
Part-Whole	35	25	9

Note. Percentages are based on n = 384 students.

Equal and Unequal Area part-whole relations as continuous quantity (Part-Whole on Equal Area and Part-Whole: Continuous on Unequal Area problems). A total of 25% treated Equal Areas as Part-Whole, but Unequal Areas as a discrete quantity (Part-Whole: Discrete), and 35% treated only the Equal Areas as part-whole, suggesting that their correct responses to Equal Areas were based on whole number counting of shaded parts and total number of parts. The pattern of results in Table 2 was similar on the posttest, although there was substantial movement to part-whole reference on both problem types: Almost all students produced part-whole relations for the Equal Area problems (92%), and 26% of students interpreted Unequal Area part-whole relations as continuous.

Relations Between Notation and Reference

We produced two sets of analyses—one cross-sectional and the other longitudinal from pretest to posttest—to document relations between notation and reference. In the cross-sectional analysis, we investigated whether students necessarily develop conventional notation and part-whole reference jointly, or whether there is some independence in the development of each. In the longitudinal analysis, we asked whether an advance in notation or reference at pretest predisposes students to acquire the other by the time the posttest was administered.

Cross-sectional analyses of notation-reference relations. To determine whether some students develop conventional forms of notation and part-whole reference somewhat independently, we cross-tabulated students' notation categories and reference categories for the pretests. Table 3 contains the results, with certain categories collapsed to focus on the primary findings. For the notation categories, we combined all categories that were unconventional. For the reference categories, we used what we regarded as the normative categories for each problem type: For the Equal Area problems, we combined all categories that were not part-whole. For the

Table 3
Percentage Distribution of Students' Modal Conventional and Unconventional Notation and Reference Categories Based on Pretest Data

	Reference	
Notation	Not Part-Whole	Part-Whole
	Equal Area problems	
Unconventional	14	7
Conventional	18	61
	Unequal Area problems	
Unconventional	34	4
Conventional	57	5

Note. For the Unequal Area problems, the Part-Whole category includes only the Part-Whole: Continuous responses.

Unequal Area problems, we combined all categories that were not Part-Whole: Continuous.

The data in the table reveal some independence between notation and reference. On Equal Area problems, 18% of the students used conventional notation without part-whole reference, and 7% used part-whole reference without conventional notation. On Unequal Area problems, 57% of the students used conventional notation without part-whole reference, and 4% used part-whole reference without conventional notation. We produced a similar cross-tabulation with our posttest data. Though student performance approached a ceiling on the Equal Area problems at posttest (88% used conventional notation to represent part-whole relations), we nonetheless found that a small percentage of students used the most advanced form of one aspect of representation but the less advanced form of the other (3% used part-whole reference but did not consistently use conventional notation and 6% used conventional notation but not part-whole reference). For the Unequal Area problems at posttest, we again produced a cross-tabulation of reference by notation and found similar results: some students used the most advanced form of one aspect of representation but a less advanced form of the other (4% used Part-Whole: Continuous reference but did not consistently use conventional notation and 55% used conventional notation but did not use Part-Whole: Continuous reference). Thus, across pretest and posttests, we found that a small to moderate percent of students use either conventional notation or part-whole reference without the other for both problem types.

Longitudinal analyses notation-reference relations. The design of our study provided an opportunity to investigate developmental relations between part-whole reference and conventional notation over time. If notation and reference influence one another in development, students' use of conventional notation *or* part-whole reference at pretest may predispose them to acquire the other at a more rapid rate than students who begin instruction without using either conventional notation or part-whole reference. To test the role of the acquisition of conventional notation, we identified students who had not created part-whole reference (on either Equal or Unequal Area problems) at the pretest stage; we then partitioned these students into two groups—those who did and those who did not use conventional notation on the pretest—and then compared their posttest performance on the reference categories for Equal Area and Unequal Area problems.

Figure 3 contains graphs of students' posttest reference categories based on their pretest notations category (Conventional vs. Unconventional). The distributions of posttest reference categories for the two groups of students are similar. A large proportion—about 70% in each group—acquired the use of part-whole relations only for the Equal Area problems; about 20% of each group used part-whole reference for both the Equal and Unequal Area problems; only a small percentage of students used no part-whole reference on posttest. Thus, we saw no evidence that accelerated acquisition of notational conventions advantages students in their subsequent development of the part-whole reference.

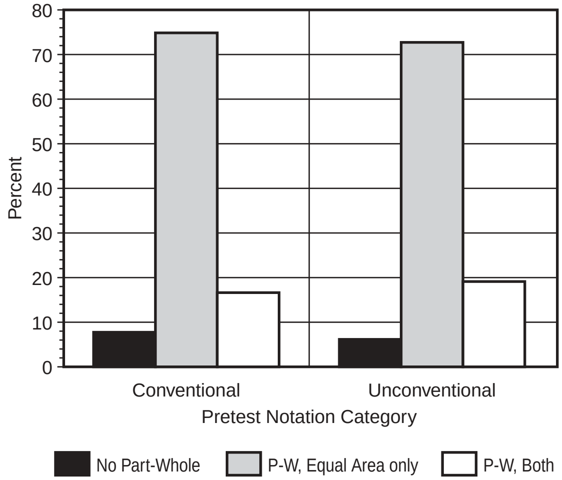


Figure 3. Percentage distribution of students' use of part-whole reference categories at posttest based on notation used on the pretest.

For the complementary analysis, we considered whether students who were accelerated in their part-whole reference at the pretest stage were more likely to acquire conventional notation by posttest. We contrasted the posttest performances of students who had not used conventional notation (on either the Equal or Unequal Area problems) at pretest, again partitioning students into two groups—those who did and those who did not show part-whole reference at pretest. One group, unfortunately, was quite small. Of the four students who consistently used part-whole reference at pretest and did not use conventional notation, all acquired conventional notation by posttest. Of the 51 students who did not use part-whole reference at pretest and did not use conventional notation, 71% used conventional notation at posttest. Since most students acquired conventional notation independent of their use of part-whole reference at pretest, we view these findings as inconclusive of the possibly supportive role of students' early acquisition of part-whole reference on the acquisition of conventional notation.

Trajectories of Change from Pretest to Posttest Focusing on the Unequal Area Problems

Our final analysis addresses students' trajectories of change from pretest to posttest as a function of different kinds of instructional conditions (High Inquiry, Low Inquiry, and Traditional). This set of analyses is important for understanding how instruction might provide differential supports for students with different kinds of understandings at the pretest. We focus only on the Unequal Area problems, which were the most challenging for students. We produced parallel analyses for Equal Area problems, but recall that student performance for both notation and reference was near ceiling levels across classrooms on the posttest, which precluded finding an effect for instructional conditions.

We present our analysis in two parts. First, to determine whether instruction differentially influenced students' trajectories of change, we conducted two log-linear analyses, one for notation and the other for reference. Second, we followed up the log-linear analysis for reference—the aspect of representation that was most difficult for students—with an analysis of students' pre- to posttest change patterns across different instructional conditions.

Log-linear analyses and results. To analyze trajectory patterns as a function of classroom practices, we conducted a log-linear analysis for notation and another for reference. For each analysis, we used three categorical variables: preinstruction response categories (T1), postinstruction response categories (T2), and instructional practice group (P). For notation, the preinstruction (T1) and postinstruction (T2) values for these variables consisted of our category scheme for notational aspects of students' representation of fractions: Conventional, Unconventional, and No Modal. The No Model category included students whose responses to the problems was not consistent. The T1 and T2 values for these variables consisted of our category scheme for referential aspects of students' representations on Unequal Area problems: Part-Whole: Continuous (P-W:C), Part-Whole: Discrete (P-W:D), a collapsed category including Part-Part and Integer hereafter denoted as PP;Int, No Modal, and Uncertain (consisting of Don't Know and Other). The instructional practice group (P) variable included three categories: High Inquiry, Low Inquiry, and Traditional. These categories for the classroom were determined for previous studies as we reported earlier in this article. For each log-linear analysis, we included main effects of T1, T2, and P, as well as all combinations of two-way and three-way interactions. Of particular interest was whether a model that included two interactions, T1 × T2 and T2 × P, would fit our data. Support for this model would indicate that students' responses on the posttest depend on their responses on the pretest (T1 × T2), and that students' responses on the posttest (T2) depend on their participation in particular classroom practice groups (T2 × P). It is the T2 × P interaction that would provide positive confirmation that our targeted classroom practice groups affect students' developmental trajectories. We found that our log-linear analysis for notation did not support the two-interaction model, thus suggesting that classroom practice group was not related to shifts to conventional notation. The proportions of students in each classroom practice group who shifted from unconventional to conventional notation were similar (67% in Traditional, 66% in Low Inquiry, and 71% in High Inquiry groups, respectively).

However, our log-linear analysis for reference revealed that the first model to fit the data included the T1 × T2 interaction χ^2 (48, N = 384) = 61, p = .10. The second model included both the T1 × T2 and T2 × P interactions, and the fit of this model was significantly better than the first χ^2 (40, N = 384) = 44, p = .32. (Note that for log-linear analyses, the greater the size of the p-value, the better the fit of the model.) Thus, the results supported both of the key interactions indicating that not only were students' uses of reference categories prior to instruction predictive of reference categories after instruction (T1 × T2), but also that the patterns of change varied in relation to type of classroom instruction (T2 × P).

Trajectories of change for reference. The results from the log-linear analyses indicated that students' pretest categories and classroom practice groups were predictive of their trajectories of change. The findings do not indicate, however, how change varied over instructional practices. To explore these patterns, we produced the four bar graphs shown in Figure 4.

Each bar graph in Figure 4 represents results from the subset of students who at pretest used a common reference category. Thus, only those results from students

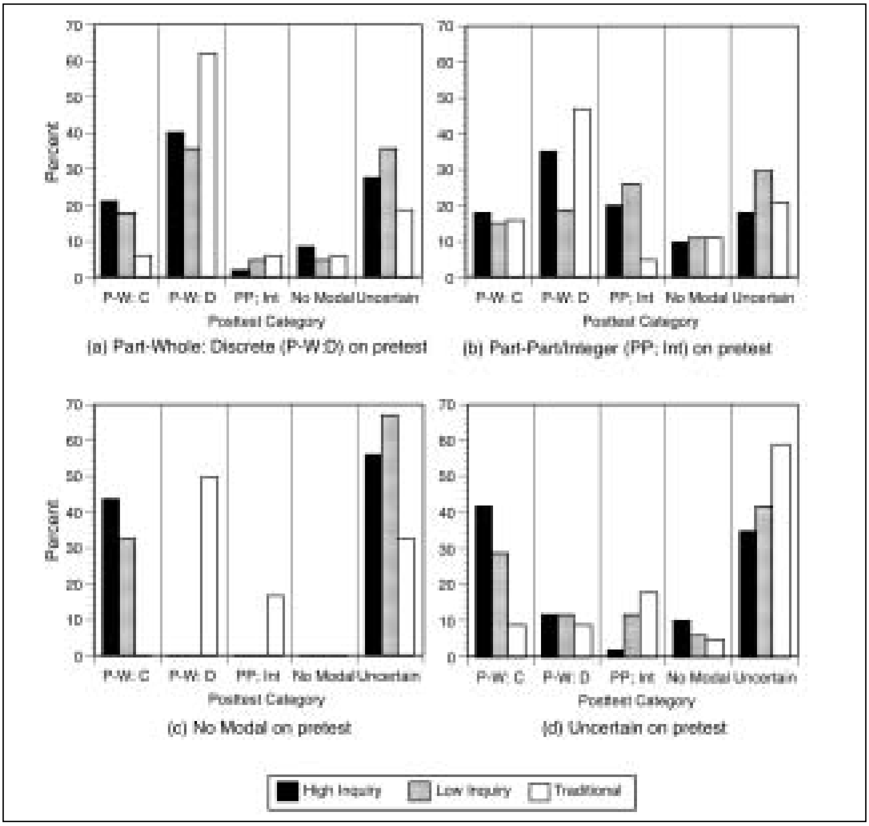


Figure 4. Percentage distribution of posttest reference categories as a function of pretest reference category and type of classroom practice.

who used the Part-Whole: Discrete (P-W:D) reference at pretest are represented in Figure 4a. Similarly, only those students who used the pretest categories of Part-Part;Integer (P-P;Int), No Modal, and Uncertain are represented in Figures 4b, 4c, and 4d, respectively. For each pretest category subset, we produced a percent distribution of results in the posttest categories as a function of instructional condition. The resulting four bar graphs then present trajectories of change patterns by instructional condition (High Inquiry; Low Inquiry; Traditional). Each graph contains results from all the students who began instruction in fractions using a specified pretest category, and the distribution contained in the graph represents different posttest categories of these students as a function of instructional condition.

Figure 4a contains results from the students who began instruction using the Part-Whole: Discrete (P-W:D) category. An inspection of the distribution of the posttest categories in this chart reveals that many students used this same category on the posttest, and this was strikingly so in the Traditional classrooms (62%) but less so in the High- and Low-Inquiry classrooms (40% and 36%, respectively). The high degree of stability in students' use of this category suggests that it provides an approach that students found to be coherent and not readily influenced by instruction. Of those students who changed categories after instruction, we found two principal directions for the shift. Some shifted to using Part-Whole: Continuous (P-W:C); the proportion of students who made this shift was greater in the High Inquiry (21%) and Low Inquiry (18%) classrooms than the Traditional classrooms (6%). Other students shifted to Uncertain (High Inquiry: 28%; Low Inquiry: 36%; and Traditional: 19%). It may be that the shift from P-W:D to Uncertain means that these students appreciated that their original choice for a reference was not an adequate representation of the Unequal Area problems, but were unclear about what the appropriate representation would be. Again, there was some evidence of the importance of inquiry instruction in that a greater proportion in the inquiry classrooms shifted to Uncertain.

Of the students who started instruction using the Part-Part;Integer (P-P;Int) category (see Figure 4b), most did not use this category at posttest, and this result was consistent across the inquiry categories (High Inquiry: 20%; Low Inquiry: 26%; and Traditional: 5%). Students who shifted from this category often adopted the P-W:D reference, particularly in the Traditional (47%) classrooms, again perhaps pointing to the apparent coherence that this category presents for students.

The No Modal category (Figure 4c) was used infrequently and with little stability: Of the students who were classified as No Modal at the pretest stage, none were No Modal at posttest. Students who shifted at posttest to Part-Whole: Continuous were much more likely to be in the inquiry classrooms (High Inquiry: 44%; Low Inquiry: 33%).

Students who began instruction using the Uncertain category (see Figure 4d) also tended to use this same category at posttest, and again, particularly so in the Traditional classrooms. Although many students in the High Inquiry (35%) and Low Inquiry (42%) classrooms were consistently uncertain from pre- to posttest, a greater portion of the students in the Traditional classrooms maintained the

Uncertain category (59%). Of those students who shifted from Uncertain to Part-Whole: Continuous (P-W:C), most were in the High Inquiry group (42%); only 29% and 9% of the students in the Low Inquiry and Traditional classrooms, respectively, shifted to P-W:C at posttest.

In sum, a shift to Part-Whole: Continuous occurred at a greater rate for the High Inquiry and at a lesser rate for the Traditional classrooms. This is the case for students who used three of the four categories at pretest. The exception is the Part-Part;Integer reference category shown in Figure 4b, where the shift to Part-Whole: Continuous occurred at about the same rate across classroom groups. Stability in trajectories was most marked for students who at pretest used Part-Whole: Discrete reference, especially for traditional classrooms, and many students shifted to this category as a result of instruction, particularly in the traditional classrooms.

DISCUSSION

Prior teaching experiments and classroom observation studies are rich with insights about changing patterns in students' use of fraction notation and reference and the role of instruction in developmental change. The methods employed in prior research, however, also have limitations. The studies were carried out in single instructional contexts with small numbers of students and made use of fraction-related tasks and activities that were not implemented in systematic ways. The analyses generated rich and provocative insights, but the generalizability of the findings may be limited. In the study reported here, we built on prior research by including a greater number of students and sampling students' performance on a set of standard fraction tasks designed to reveal developmental relations between students' uses of notation and references for those notations. Our design allowed for systematic analyses of students' representation of part-whole relations on Equal Area and Unequal Area fraction problems, notation-reference relations on these problems, and the role of instruction in patterns of developmental change. We discuss each below.

Relations Between Notation and Reference

Our findings show that students' knowledge of conventional notation and part-whole relations can develop somewhat independently. From our analyses of students' performance at pretest (replicated at posttest), we found that some students used unconventional notational forms to refer to part-whole relations, and some students used conventional forms to refer to relations that were not part-whole. These patterns varied with problem type; for example, many students referred to part-whole relations only on Equal Area problems. Our analyses of notation-reference relations from pretest to posttest with instruction in between showed that students' use of conventional notation at pretest did not lead to any advantage in their development of part-whole reference at posttest. Together, these findings suggest that teaching students notational skills may not support the development of reference, at least as we define it here. Because of limitations in pretest profiles of our sample, we could not put the reciprocal relation to the test—that knowledge of part-whole reference at pretest leads to an accelerated knowledge of conventional notation.

These findings have important implications for classroom practice. As Ball (1993) and Mack (1995) have argued, teachers need to guide students in investigating relations between fractions concepts and fractions notation in multiple contexts. The "whole" may be a continuous or a discrete quantity; the continuous quantity may be a linear, area, or volume model. Consistent with the NCTM's *Principles and Standards for School Mathematics* (2000), students need to make connections among mathematical representations both of the quantities and the notations for those quantities. And, as teachers monitor their students' progress, they must recognize that a child's understanding of either notation or part-whole relations does not ensure the other.

Role of Instruction in Developmental Shifts in Notation and Reference

Our study contributes additional findings on the role of inquiry-oriented instruction in student learning (e.g., Carpenter et al., 1989; Fennema, et al., 1996; Hiebert & Wearne, 1996; Saxe et al., 2001; Saxe et al., 1999; Riordan & Noyce, 2001). Across these studies, researchers cite fairly consistent evidence that inquiry instruction, when compared with traditional instruction, is associated with greater gains in students' conceptual understanding and problem solving. Our results are consistent with these prior studies. Students in inquiry classrooms made greater progress than those in traditional classrooms within their representation of part-whole relations on the Unequal Area problems. That pattern was stronger in the High Inquiry classrooms where teachers were more likely to elicit students' thinking about fractions and engage students in mathematical discussions of part-whole relations. These findings are promising but do not address the concern that inquiry instruction provides too little attention to facts and procedures. Indeed, to address this concern, advocates of inquiry instruction must demonstrate that not only do students profit at a greater rate with inquiry methods on tasks that assess conceptual understanding, but also that students' mastery of facts and procedures is not disadvantaged by inquiry methods. This is precisely what we found: Our findings show no difference between inquiry-based and more traditional classrooms in the rate of improvement in students' performance on notational aspects of representation. Thus, our findings support the value of inquiry instruction in students' learning of fractions notation and reference.

Part-Whole: Discrete was a common and fairly stable form of reference from pretest and posttest; further, other students who used a Part-Part;Integer approach at pretest often shifted to Part-Whole: Discrete at posttest. The stability and frequency of this reference category have two likely sources. On the one hand, Part-Whole: Discrete forms of reference can be created as an extension of whole number interpretations of parts of areas; on our problems, a child need only count the gray parts and all of the parts to produce a part-whole representation. On the other hand, classroom practices may either explicitly or inadvertently support students' use of Part-Whole: Discrete if the area models presented contain stereotypical same-sized parts, and if teachers do not engage students in analytic discussions of size relations between parts and wholes. After all, Part-Whole: Discrete approaches are successful for areas partitioned into equal parts, a common representation when introducing area models of fractions.

Our analyses of the role of classroom practice support these arguments. Our classroom scales captured the extent to which teachers engaged students in discussions of part-whole concepts in ways that built on students' understandings. In traditional classrooms, these ratings were lower, as we might expect in settings where textbooks emphasize fractions vocabulary and skills. Thus, a majority of students in traditional classroom who began instruction with Part-Whole: Discrete reference—one that extends whole number operations to problems of fractions—ended with this same approach. Further, those students in Traditional classrooms who shifted from Part-Whole: Discrete often moved to uncertainty, and many students who were categorized as Uncertain at pretest remained so at posttest. Students' use of the Uncertain category on the Unequal Area problems may indicate that students know that Part-Whole: Discrete is an inappropriate form of reference on this problem, but also lack knowledge of the appropriate reference.

Concluding Remarks

A child's representation reflects the coordination of knowledge of notational conventions and particular kinds of part-whole relations. In the elementary grades, both notation and reference are developing, but not always fully in concert. As students begin to construct understandings of fractions concepts and notation, their knowledge is rooted in whole number, such that a "correct" notation may belie immature understandings of fractions. Yet at other times, students may produce unconventional representations that refer to some sophisticated intuitions about fractions. This study produced a fine-grained developmental analysis of notation-reference relations that contributes to our understanding of representation in the domain of fractions. We also produced an analysis of the role of classroom practice, showing advantages of inquiry approaches that focus on fractions concepts and representations, and build on student understanding.

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