From the Field to the Classroom: Studies in Mathematical Understanding

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Children's social and cultural lives are interwoven with their developing mathematical understandings in fundamental ways. In their efforts to document children's understandings, researchers have often sidestepped analyses of sociocultural processes in children's mathematics. In this chapter, I sketch a research framework in which dimensions of daily life are elevated to a central target of analysis. I show that research guided by the framework is useful for informing the creation of classroom practices for children's mathematics learning, and that the research framework, in turn, is useful for understanding children's learning in such practices.

THE RESEARCH FRAMEWORK

The guiding assumption of the framework is one consistent with constructivist treatments of various theorists including Piaget (1963b, 1977) and Vygotsky (1978, 1986): Individuals construct novel understandings as they attempt to accomplish goals rooted in both their prior understandings and socially organized activities. To date, however, these treatments have not been adequately extended into systematic frameworks for the analysis of the interplay between culture and cognitive development. For Piaget, sociocultural processes were largely unanalyzed. For Vygotsky, such processes, although of central importance, were not incorporated into a systematic research approach (for a more extensive critique, see Saxe, 1991).
The research approach I summarize is framed around the central construct of goals. Cognitive goals emerge in individuals’ daily participations in cultural practices. In attempting to accomplish these emergent goals, children generate new knowledge linked to social and cultural life.

There are three components within the framework. The first component targets for analysis processes of goal formation in cultural practices. The second is concerned with processes of cognitive development linked to children’s efforts to accomplish emergent goals. The third component is concerned with the interplay between developments across practices: the way cognitive developments linked to children’s participation in one practice may be appropriated and specialized in children’s efforts to accomplish emergent goals in another.

ILLUSTRATIONS OF THE RESEARCH FRAMEWORK

My research in Papua New Guinea and Brazil provides illustrations of the approach.

Economic Exchange: An Illustrative Practice-Based Study with the Oksapmin of Papua New Guinea

The Oksapmin, a cultural group numbering from 6,000 to 8,000 members, populate a remote region of Papua New Guinea’s West Sepik Province. First Western contact with the Oksapmin was made by an Australian patrol in 1938–1940. Since first contact, the group has remained isolated; the only access to the area is through small aircraft or trek by foot. For subsistence, Oksapmin traditionally harvest taro and sweet potato with slash-and-burn methods and hunt for small game with bow and arrow. One feature of Oksapmin life that made study of the community particularly interesting from the perspective advanced here was the character of Oksapmin’s number system—a system that has no base structure and in which body parts are used to signify numerical relations. To count as Oksapmin do, one begins with the thumb on one hand and follows a trajectory around the upper periphery of the body down to the little finger on the opposite hand; to express a particular number, one points and/or states one of 27 conventionally defined body part names. The system is depicted in Fig. 15.1.

Component 1: Emergent Numerical Goals in Oksapmin’s Daily Practices. The mathematical goals that emerge in the everyday lives of Oksapmin people and Westerners differ markedly from one another. In contrast to the West, in traditional Oksapmin life, numerical problems
were not arithmetical. Rather, the numerical problems of daily life involved such activities as making a count and communicating about the number of pigs in one's possession, measuring the lengths of string bags (a ubiquitous artifact), and communicating about ordinal positions such as points on a path. With an emerging money economy brought through Western contact, many Oksapmin began to address arithmetical problems, forming mathematical goals unlike those they had generated previously.

To understand the interplay between social and developmental processes, I found it useful to conceptualize emergent goals with reference to a four-parameter model (Saxe, 1991). These parameters are depicted in Fig. 15.2. They include: (a) the activity structure of the practice, (b) the social interactions in which goals become modified and take particular forms (through assistance and negotiation), (c) particular sign forms and cultural artifacts, and (d) the prior understandings that individuals bring to bear on practices. I illustrate each of these parameters and the way they are interwoven with the emergence of individuals' goals, using economic exchanges at Oksapmin trade stores as an example.

The activity structure of a cultural practice consists of task-linked motives and recurrent phases that must be accomplished in the practice. For instance, to participate in economic exchanges in Oksapmin trade stores, one must accomplish a purchase. In a purchase, a principal motive may be to acquire as many desired goods as possible while sacrificing as little money as possible, and arithmetical goals that emerge in exchanges may be guided by this economic motive. Issues of small change in a transaction, although significant from the point of view of a normative mathe-
mathematics, may be insignificant from the point of view of an individual’s economic motives linked to the practice’s structure.

Second, social interactions between participants in practices may further influence the character of the goals that individuals address. In the economic exchange in an Oksapmin trade store, a trade-store owner may help the customer with a computation in any number of ways, modifying the kinds of arithmetical goals a customer addresses. Such assistance permits customers of varying degrees of competence to participate in the practice. In the case of the trade store, assistance may be in the form of helping a customer add coins, completing a subtraction problem in the computation of change, or helping to clarify an arithmetical problem. Regardless, problem-linked social interactions often are interwoven with the goals and subgoals that emerge and that individuals accomplish in the practice.

Third, cultural forms that have emerged over the course of social history, such as historically elaborated sign forms like the Oksapmin indigenous body part counting system and cultural artifacts like a particular currency system, also figure into the goals that emerge in cultural practices. For instance, the body part counting system leads Oksapmin to such goals as adding a bicep to a forearm, a representation of an addition problem that both constrains and supports the construction of particular kinds of subgoals for a computational solution. Similarly, the particular denominational structure of a currency system also influences the particular values addressed in activities and the subgoals that emerge in problem solving. For instance, in the Oksapmin case, the structure of the currency consists of a base of 20 (linked to the original currency system to penetrate the area—the base-20 system of Australian shillings and pounce), in which 20...
ten toa coins are equivalent to one 2 kina note. This system has led to particular problems when adding currency units that differ in some fundamental ways from those that emerge with a base-10 currency system.

Fourth, the prior understandings that individuals bring to bear on cultural practices both constrain and enable the goals they construct in practices. In the case of the Oksapmin, we find that individuals who had different levels of experience participating in the money economy bring to bear different arithmetical understandings on practice-linked problems, and consequently their goals differ. Thus, an Oksapmin with little experience in the money economy typically conceptualized a purchase in terms of a multiple items of merchandise for multiple units of currency exchange. Thus, the goal was to produce an appropriate one-for-one or many-for-one correspondence. In contrast, individuals with greater expertise conceptualized problems as arithmetical ones, summing the total cost of items of merchandise.

In summary, the four-parameter model is a basis to understand the interpenetration of the cognizing activities of the individual and the enculturating dimensions of social practices in goal formation. In participating in the practice of economic exchange, Oksapmin participate in an activity structure in which particular arithmetical problems must be accomplished that have implications for goal formation (Parameter 1). Further, in the activity, Oksapmin store owners’ and customers’ numerical goals become interwoven with symbolic forms and artifacts like the indigenous number system and the currency system (Parameter 2), and these goals take form and shift in customer–store owner interactions (Parameter 3). Finally, the arithmetical understandings that store owners and customers bring to bear on the practice are deeply interwoven with the ways in which they create and recreate arithmetical goals (Parameter 4).

Component 2: Form-Function Shifts in Cognitive Development. Component 2 of the approach consists of a conceptualization of cognitive developmental processes that builds on analyses of emergent goals. In this cognitive developmental model, I examine the interplay in development between cultural forms (like number systems, currency systems, and other forms of social conventions) and cognitive functions (like enumeration and arithmetic) that are intrinsic to individuals’ efforts to accomplish emergent goals in practices. Further observations in the Oksapmin community provide a good illustration of the interplay between form and function.

As the Oksapmin’s level of participation with economic exchanges involving currency increased, the Oksapmin addressed new goals that

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1The form-function model as applied to cognitive development is an extension of Werner and Kaplan’s (1962) original treatment.
required the addition and subtraction of values—goals that were previously foreign to them. In this process, the Oksapmin gradually structured new functions for the body counting system. The shift is one in which the body system, a form that traditionally only serves an enumerative function for the Oksapmin, became a form that served an arithmetical function.

To document form-function shifts in the Oksapmin’s problem solving, I interviewed Oksapmin who had different levels of participation with economic transactions. The interviews showed that, with increasing levels of participation, the Oksapmin appropriated and specialized the indigenous cultural form—the body system—to address the emergent problems in exchange. In this process, the function that the indigenous body part form serves gradually shifted. Consider four approaches to the solution of 9 + 7 coins that emerged with increasing participation in the money economy (for an extended treatment, see Saxe, 1982, 1991) depicted in Fig. 15.3 and explained next. The approaches shown in the figure are distinct cognitive or strategic forms, each linked to the indigenous cultural form of the 27 body part counting system.

With initial participation in the money economy, the Oksapmin first attempted to extend the body counting cognitive form, as it was used to serve enumerative functions in traditional activities, to accomplish arithmetical tasks that emerged in economic transactions. However, this direct extension was not adequate to accomplish arithmetical solutions, and it was not even clear that Oksapmin with little experience treated the task as one that involved the cognitive function of arithmetic. In these preliminary efforts, the Oksapmin attempted to count the sum with a

![Fig. 15.3. Four strategies for solving 7 + 9 = ?](image)
"global enumeration strategy"—a counting strategy used for everyday enumerative functions awkwardly applied to the arithmetical problems. In this strategy (depicted in Fig. 15.3a), to solve $9 + 7$ coins, an individual begins with one term of the problem (thumb (1) to forearm (7)) and then continues to count the second term (9) from the elbow (8). Because the problem of $9 + 7$ coins seemed to be understood as an enumeration rather than an addition, individuals did not recognize the need to keep track of the addition of the second term onto the first term, and they typically produced an incorrect count of body parts, not keeping precise track of their count.

With greater economic experience, there was a shift in the use of the body system, a shift in which the body counting form was adapted to serve a new function. Oksapmin with greater economic experience often made a labored effort to restructure their prior global counting strategy in such a way that one term is added onto the other in problems like $9 + 7$ coins. In one example of this strategic form (depicted in Fig. 15.3b), individuals again enumerated one term (thumb (1) to forearm (7)), but now, as they enumerated the second, they make efforts to keep track of their enumeration. Thus, the elbow (8) was paired with the thumb (1), the bicep (9) was paired with the index finger (2), and so on until the ear on the other side (16) was paired with the bicep (9), yielding the answer. Thus, in this initial extension of the body system to accomplish the arithmetical problem, the body parts began to take on a new function—keeping track of the addition of one term onto another.

With higher levels in the sequence, Oksapmin specialized their use of the body part counting system to serve distinctly arithmetical functions. With greater experience, individuals efficiently use the name of one body part to refer to another in a "body substitution strategy" (Fig. 15.3c), rather than establishing physical correspondences between body parts as they did previously. To solve $7 + 9$, the elbow (8) is called the thumb (1), the bicep (9) is called the index finger (2), and so on until the ear on the other side (16) is called the bicep (9). The result is a more rapid computational process—one in which body part names are differentiated from the names of body parts themselves.

Cognitive forms that are distinctively specialized to serve arithmetical and not enumerative functions were most frequently displayed by trade-store owners—individuals who had the most experience with economic transactions. Here we see the incorporation of a base-10 system (linked to the currency) as an aid in computation. To solve the same problem of $9 + 7$, trade-store owners often displayed a procedure termed a halved-body strategy (Fig. 15.3d). With this strategic form, individuals use the shoulder (10) as a privileged value. In their computation, they may represent the 9 on one side of the body as bicep (9) and 7 on the other side of the body as
forearm (7). To accomplish the problem, a trade-store owner might simply remove the forearm from the second side (the seventh body part of 7) and transfer it to the first side, where it becomes the shoulder (the 10th). He then reads the answer as 10 + 6 or 16.

The form-function shifts in the Oksapmin case reflect the Oksapmin's efforts to accomplish goals that emerge in their everyday life. In attempting to accomplish these goals, the Oksapmin use prior body part strategies initially specialized to serve the cognitive function of enumeration. In this process, the function of the body system shifts, and the original form used to accomplish enumerative functions undergoes a progressive development. In this process, the Oksapmin create, over time, progressively more specialized and sophisticated uses of the indigenous number system, a process of specialization that is deeply interwoven with the construction of the new arithmetical function.

Component 3: The Interplay Between Cognitive Developments Across Practices. The focus of my Oksapmin research was principally on shifting goals that emerged as a result of new practices (Component 1) and the interplay between cognitive form and function within the practice (Component 2). To offer a more complete analysis of the interplay between culture and cognitive development would have required an analysis of the interplay between emergent goals and cognitive developments across practices. In turning to investigations with child candy sellers in Brazil, I focus on the practice of selling candy, as well as the interplay between math learning in the selling practice and in school.

Brazilian Child Candy Sellers and the Interplay Learning Across Practices

The site of the candy-selling investigation was Recife, an urban center in Brazil's Northeast. As in the case with the Oksapmin research, I focused on the development of mathematical understandings linked to economic exchange. However, the social and cultural conditions differed markedly from Oksapmin. Indeed, Recife is a bustling city in which many poor children attempt to generate an income through street selling. Child vendors offer a wide range of merchandise that shift with market conditions including fruit, puffed wheat, vegetables, and, of course, candy. Again, my concern was to extend the three-component approach to understanding the interplay between sociocultural and cognitive developmental processes.

2The Oksapmin strategy descriptions are excerpted from Saxe (1991).
Component 1: Emergent Goals in Selling Candy. The four-parameter model (refer to Fig. 15.2) provided a basis to study how sociocultural processes were interwoven with candy sellers’ construction of arithmetical goals as they plied their trade. As in the case of the Oksapmin, my concern was to understand the process of goal formation as it is interwoven with the structure of the practice, artifacts, patterns of interaction, and the understandings that individuals bring to bear on the practice.

The activity structure of the candy-selling practice is contained in Fig. 15.4. The figure depicts an entrepreneurial activity that has the cyclical organization of buying merchandise (in the purchase phase), markup (in a prepare-to-sell phase), retail selling (in the sell phase), and considering best wholesale buys (in a prepare-to-purchase phase). Mathematical goals emerge in each phase of the practice. However, for illustrative purposes, I only consider the sell phase.

The sell phase is one element of the practice’s larger entrepreneurial structure (Parameter 1). In this phase, children offer their goods to potential customers, and interchanges with customers are often guided by the concern to accomplish a profitable transaction. The profit motive linked to the structure of the practice has implications for particular sell-phase goals. For instance, in exchanging goods for currency, arithmetical goals may emerge linked to making change or to conducting a profitable transaction.

Two principal artifacts/conventions (Parameter 2) frame children’s goals in the sell phase. First, Brazilian currency is the medium for economic exchange, and arithmetical problems that emerge are interwoven with this artifact. Brazil has had a long history of inflation, and frequently used denominations of currency represent very large numerical values. Sellers operate on such large values in their economic computations. Second, a pricing convention has emerged in the practice (setting prices in ratio form like three bars for 1,000 cruzeiros). The ratio convention frames a many-to-one retail exchange between seller and customer, and thus constrains the nature of the mathematical subgoals that emerge in seller–customer transactions. The ratio convention also constrains sellers’ markup computations, since sellers must compute their retail prices in terms of the convention.

Social interactions (Parameter 3) are fundamental to the emergence of particular kinds of goals in the sell phase. It is in seller–customer transactions that arithmetical problems are formulated and reformulated. Further, sometimes, depending on the interaction, these problems may be quite complex. For instance, in making a purchase, a customer may bargain with a seller, which may lead the seller to some rapid recomputations of markup, again provoking the reemergence of mathematical goals linked to markup.
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Analyses of the interviews revealed that, despite their lack of schooling and poverty level subsistence, most sellers developed mathematical knowledge and problem-solving approaches that were quite remarkable. For instance, although unable to read or write numerical values with proficiency, even the youngest sellers developed the ability to represent large numerical values using currency. They could identify currency notes by value and they knew ordinal and multiplicative relationships between currency units (e.g., five Cr$200 bills is equivalent to one Cr$1,000 bill).

With age, sellers, more so than their nonselling peers, increasingly showed the ability to solve complex arithmetical problems with very large values (linked to the large currency denominations), ratio-comparison problems (linked to the price-ratio conventions), and markup problems (linked to pricing activities). Like the Oksapmin study, in the case of the candy sellers I found that, with development, sellers showed evidence that forms first appropriated to serve more elementary numerical functions were reorganized to serve more complex functions. For instance, young sellers quickly developed the ability to identify appropriate currency denominations in seller–customer exchanges. Currency identification served an important function in the sell phase, enabling a young seller to adequately complete a transaction with a customer. With age, a shift in the cognitive function that currency serves for sellers was observed. Rather than serving solely as a medium of exchange, sellers now operate on currency values, manipulating them in performing arithmetical computations.

Like the Oksapmin research, Components 1 and 2 provide a portrayal of the emergence of goals in a single practice and the individual's development of practice-linked knowledge. Component 3 is concerned with the interplay between development across practices, as children bring to bear cognitive developments linked to one practice to accomplish emergent goals linked to others.

In the study of candy sellers, I investigated the interplay between the mathematical developments that children were forming in the candy-selling practice and at school. To accomplish this, in one set of analyses I contrasted sellers with different levels of schooling on tasks used to assess their candy-selling-linked mathematics. In a second set of analyses, second- and third-grade sellers and nonsellers were contrasted in the way they solved standard paper-and-pencil computational tasks linked to school. The two sets of analyses showed evidence of a complex interplay between learning across practices. Consider evidence that children were making use of the mathematics of the selling practice in addressing problems that were linked to school.
I noted that in their practice, candy sellers operate on bill values to solve arithmetical problems. They typically do not use algorithms dictated by a teacher, inventing, instead, their own ways to achieve coherent solutions. For instance, in adding a collection of 15 bills, a seller might organize them into convenient values, grouping, for instance, five Cr$200 bills, calling it Cr$1,000, and then summing these intermediate units into a grand total. Similarly, they typically do not use paper-and-pencil computations in markup, but rather structure strategic forms that reflect a conceptual understanding of the mathematical relations entailed in markup solutions. Such knowledge forms show a flexibility in combining and recombining terms—a flexibility that may not be well nurtured through the drill-and-practice instruction that often occurs in school.

I suspected that sellers may appropriate the flexible bill value knowledge to accomplish arithmetical problems in school. To test this idea, I interviewed sellers and nonsellers using tasks that involved paper-and-pencil arithmetical problems like the ones that they solve in classrooms. Children’s accuracy and problem-solving strategies were coded. We found marked differences in the accuracy and strategies of sellers and nonsellers. Sellers more frequently achieved accurate solutions, and the source of their accurate solutions were regrouping strategies—strategic forms analogous to the ones that they used on bills. In contrast, nonsellers’ accurate solutions were more typically generated with algorithmic solution strategies. Thus, we do see evidence that knowledge generated in the selling practice finds its way into children’s efforts to accomplish goals that emerge as children attempt to accomplish mathematical problems linked to school.

FROM THE FIELD TO THE CLASSROOM

The results of the studies with the Oksapmin and candy sellers show that, quite apart from formal instruction, considerable math learning can occur in informal, out-of-school contexts by population groups that one would not expect to distinguish themselves in school. These findings take on special educational significance when we consider the reports of low levels of achievement and interest in school math learning in many underclass children in our own country (Alexander & Entwisle, 1988; Coleman et al., 1966). How might this practice-based research inform mathematics instruction in classrooms in the United States? I believe that there are two avenues.

First, by building on children’s out-of-school mathematics, teachers may be able to link the mathematics curriculum with children’s own purposive activities for which they have some investment. For instance, in the case of

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Brazilian candy sellers use a series of classroom exercises to solve various types of problem, and strategies to solve these problems, the teacher may be in relation to target.

In fact, out-of-school activities are likely to be cultural and ethnically diverse and may involve shopping at stores or in one’s neighborhood. Despite the diversity, it is important to take on the role of the teacher in planning the lessons that may involve rich problems.

A second approach to mathematics instruction is to provide a variety of classroom activities that have been shown to be effective. This may involve designing lessons that are relevant to children’s experiences, as in the problem classroom. In developing lessons that are related to children’s everyday experiences, one might consider the following:

1. *Mathematics*: Oksapmin and candy sellers use a variety of practices and strategies to solve problems. These strategies can be used in mathematics classrooms as well.

2. *Mathematics*: In both candy sellers and Oksapmin, learning was not limited to academic instruction but occurred in a variety of contexts and situations.
Brazilian candy sellers, a teacher and children could exchange expertise. In a series of classroom activities, Brazilian teachers could elicit the children’s various approaches to markup, extend these approaches to a wider range of problems, and try to understand correspondences between different strategies to solve this class of problems. In the course of these activities, the teacher may introduce ways to formalize these problems and solutions in relation to targeted instructional objectives.

In fact, out-of-school practices in which mathematical problems emerge are likely to be common experiences for children across a wide range of cultural and ethnic groups. Such practices may vary markedly. Some may include trading baseball cards, others working with origami, or still others shopping at stores or through catalogs. Thus, one may envision a general approach to instruction in which teachers use children’s areas of expertise as starting points to organize classroom instructional activities.

Despite the merits of such an approach, I believe that it would not work well for many classroom contexts. The approach requires the teacher to take on the role of the anthropologist, structuring class activities and lessons that mesh with often a heterogeneous group of children—children who may be engaged with widely different practices, only some of which may involve rich mathematical content.

A second approach is related to the first, but it addresses the problem of classroom heterogeneity: Teachers can engineer a classroom practice that has properties of the daily practices involving mathematics in which many children show sustained engagement. As a practice-linked mathematics emerges, teachers can then help children link their developing understandings with mathematical domains targeted for formal instruction, as in the prior approach. I have been pursuing this second approach in ongoing research in an underclass, ethnically diverse group of inner-city school children in the Los Angeles area.

In developing a classroom practice, I have been guided by the following analysis of some features of economic exchange in Oksapmin and candy selling that appeared to promote mathematics learning in these practices. Consider the following:

1. **Mathematics was not a target of instruction.** In both the case of Oksapmin and candy sellers, individuals were engaged in practices in which learning mathematics was not the goal of participation. Indeed, if instruction occurred at all, it was in the context of problem solving and generally took the form of assistance.

2. **Mathematics learning served the accomplishment of pragmatic objectives.** In both candy selling and in Oksapmin economic exchange, mathematics learning was not an end in itself, but was interwoven with the purposes of acquiring needed goods or earning money.
3. *Artifacts shaped the form of emergent mathematical problems.* Artifacts, like the currency systems, were interwoven with the mathematical problems with which individuals were engaged; further, these artifacts often became vehicles that individuals used to mediate problem solving.

4. *Emergent problems displayed a range of complexity levels.* Problems of multiple difficulty levels emerged in practices. In the case of candy selling, some problems were mathematically straightforward, such as producing a one-for-one exchange of a bill value for candy bars. However, others were more complex involving computations of markup.

5. *Individuals played an active role in problem formation.* Individuals often had some role in the complexity of the problems with which they were engaged. For instance, a Brazilian candy seller might sell for only a single ratio, and thereby circumvent problems of ratio comparison. Or a traditional Oksapmin might rely on the aid of a trade-store owner in accomplishing an exchange. In this way, individuals of multiple ability levels were able to participate.

6. *The solutions of mathematical problems were valued for their coherence, not for the correct use of rigidly prescribed procedures.* In solving emergent problems, individuals were not principally concerned with recalling a recipe for solution dictated by a teacher. Rather, emphasis was on finding coherent solutions to emergent problems.

In my inner-city work, my students and I have engineered a classroom practice that makes use of the insights sketched previously. Our investigations of the children's developing practice-linked and school-linked mathematical understandings are guided by the research framework I have used in my previous work.

**The Classroom Practice**

My students and I constructed a board game (see Fig. 15.5) in which children assume the roles of treasure hunters in search of gold doubloons (gold painted base-10 blocks in denominations of 1s, 10s, 100s, and 1,000s). The game is a thematic one. In their search to acquire the most gold, children sail small ships around six islands (see Fig. 15.6) through rolls of a die. At each island, children enter a port where, with their gold, they may purchase supplies (e.g., shovels, spy glasses, snake repellent) whose prices are posted in ratio form (e.g., at Snake Island: one shovel

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3The research group included Joseph Becker, Teresita Bermudez, Steven Guberman, Marta Laupa, Scott Lewis, David Niemi, Mary Note, Pamela Paduano, and Christine Siarczak.
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FIG. 15.5. The Treasure Hunt game, with arrows pointing to players' treasure chests (where gold doubloons [base-10 blocks] are kept) and gold registers (where the value of players' gold doubloons are represented).

for four doubloons, two for seven doubloons, or three for nine doubloons). After their purchase (or decision not to purchase), players draw colored island cards that direct them to a particular geographical region where players' supplies take on functional significance. For instance, at Snake Island, depending on the draw, players may land on an orange region that states: "If you have a shovel and a treasure map, you can dig up a golden snake worth 55 doubloons. Take your money from the bank." Alternatively, the brown region states: "If you have a lantern, poisonous snakes will avoid you. If you don't have a lantern, pay 17 doubloons to hire a guide." Having accomplished the operation linked to the region, the player must adjust his or her gold register (see Fig. 15.5) to reflect the amount of gold in his or her treasure chest in terms of the standard orthography. Then, the turn shifts to the partner.

In the design of the game, we made an effort to instantiate the six properties described earlier to support children's construction of a rich mathematical environment in play.
1. There was no explicit math instruction. We did not offer children formal instruction in mathematics. In fact, aside from an introductory “how to play the game” lesson, children were asked to use one another, as opposed to a teacher, as resources in solving problems with which they were having difficulty.

2. Math learning occurred as children accomplished game-linked objectives. In Treasure Hunt, mathematics learning was not an end in itself. Rather, mathematical problems emerged in play, and children, we believed, would be motivated to solve them.

3. Base-10 towers: We selected the base-10 block system because its physical manipulatives align with the base-10 block system in order to allow children to conceptualize each digit in a multi-digit number.

4. Treasure structured towers: The children in the form of games, which they determined to compare

5. Treasure of mathematics: Treasure Hunt emerged at the following levels: (1) and so on. This involves going to purchase a digit for a given source. It another to problems.
would be motivated to find solutions to these emergent problems. These problems were not merely appended to play, but were intrinsic to the process of making purchases and acquiring gold.

3. **Base-10 blocks were educationally significant artifacts used in exchanges.** We selected the base-10 block gold as the principal artifact of exchange. Various educators and researchers have argued that base-10 blocks are a symbolic form in which the conceptual meaning of arithmetical transformations is more transparent than in the standard number orthography (see, e.g., Dienes, 1964; Resnick & Omanson, 1987). For instance, in the base-10 block analogue to borrowing in subtraction, a child exchanges a higher denominational block for 10 units of the lower denominational block in order to subtract. This equivalence transformation is supported by the visual size relationships of the blocks (the higher denominational block is 10 times the size of the lower denominational block) and the physical markers. By conceptualizing and framing arithmetical problems in the medium of base-10 blocks, we expected children to structure a rich schema for understanding arithmetical relationships that could be used to conceptualize our more opaque number orthography. Further, requiring children to report their value in gold in the orthography seemed to be a way of promoting such insight.

4. **Treasure hunters played a significant role in problem formation.** We structured the game in such a way that the children played an active role in the formation of the problems. For instance, children determined the amount of goods they would purchase (they need not purchase any), they determined whether to compare ratios, and they determined whether to compare prices when shopping.

5. **Treasure hunters could engage themselves with problems at varying levels of mathematical complexity.** We created the conditions for problems to emerge at multiple levels of complexity. For instance, if a child had 1,008 doubloons in the form of one 1,000 block and eight 1 blocks [1 (1000), 8 (1)] and sought to purchase supplies for 6 doubloons, the child’s solution involves merely the payment of six 1s. However, should the child wish to purchase supplies that cost 59 doubloons, the emergent problem would be considerably more complex, requiring gold trades of larger for smaller denominations from the bank (the analogue to the “borrowing” or “regrouping” algorithm).

6. **Norms for problem solving were not solution recipes.** Finally, we offered no prescriptions for procedures to solve problems. When children asked for a solution, we referred children back to their partners as a resource. It was largely left up to children in communication with one another to structure coherent means of accomplishing the emergent problems.
The Research Approach Applied to the Classroom Practice: Does Play of Treasure Hunt Lead Children to Structure Rich Mathematical Environments and New Understandings?

In our research with Treasure Hunt, we created three groups of children by random assignment. Two groups were game players: One of these groups was organized in dyads of similar ability levels, and the other into dyads of mixed ability levels. Our focus was on comparisons between the patterns of problem formation and assistance interactions between these mixed and same-ability groups. The third group served as a control, and was engaged in other academic activities during game play periods. These nongame players were children drawn from the same classrooms and of the same abilities of the children in our experimental groups.

The Treasure Hunt study consisted of two principal phases. During the first phase, the two dyadic groups played Treasure Hunt twice weekly over a 2½ month period for about ½ hour per session. We videotaped the first and last session of play for each dyad. We then posttested all children on game-linked and school-linked math problems. In the second phase, we designed whole-class lessons. We helped children understand the algorithmic mathematics of their classroom in terms of base-10 manipulatives.

Our analyses of data generated from the Treasure Hunt project is currently underway. Each of the three components of the research framework serves to drive the analyses. I illustrate some of the analytic directions next.

Analyses of Emergent Goals in Play (Component 1)

As in the studies with the candy sellers and the Oksapmin, we analyze emergent goals in Treasure Hunt with reference to the four-parameter model previously presented in Fig. 15.2.

With regard to the structure of the game (Parameter 1), we are concerned with how players' emergent mathematical goals are linked to the organization of the game and players' principal motives. An inspection of the rules reveals that each turn consists of a five-phase structure depicted in Fig. 15.7, consisting of the challenge, rent, purchase, region, and check phases. Across phases, children are guided by the motive to acquire more gold, although in each phase children often address phase-specific mathematical goals. For instance, in the challenge phase, players compare their partners' quantity of gold (base-10 block form) with their partners' representation of gold in their gold registers (orthographic form) to determine the adequacy of the correspondence...
FIG. 15.7. The 5-phase, turn-taking structure of Treasure Hunt.

(players gain gold if they produce an appropriate challenge). In the rent phase, players may need to subtract gold from their treasure chests to pay rent if they have landed on an island containing their partners’ forts or castles. In the purchase phase, players may compare price ratios in shopping for needed supplies and tally up the items for purchase, and then subtract from their treasure chest to pay the appropriate amount. In the region phase, players may need to add or subtract from their gold registers, depending on the particular region message and the players’ available supplies. Finally, in the check phase, players alter their gold register to reflect the gold in their treasure chests.

The particular artifacts (like base-10 blocks) and conventions (Parameter 2) that are features of the practice figure prominently into children’s mathematical goals. To illustrate, consider the purchase phase. Motivated by a concern to acquire more gold, players may decide to make a purchase at the trading post. In this purchase, children’s mathematics becomes intertwined with properties of the base-10 blocks; that is, to perform a subtraction of 1,004 – 27 requires the construction of different subgoals and operations when the problem is presented in gold blocks (e.g., one 1,000 block and four 1 blocks take away a value of 27) than when it is presented in the form of the standard orthography (1,004 – 27 in vertical format).

The particular social interactions (Parameter 3) that are intrinsic to play occasion the emergence of or modify mathematical goals. For instance, with a treasure chest containing 1,004 doubloons [1(1,000s) 4(1s)], a player may want to spend 27 doubloons on treasure chests and maps. The player may get stuck, unable to obtain the needed 27 doubloons from either the 1,000 block or the four 1s. At this point, the partner may suggest trading the 1,000
block for 100s, and the suggestion may lead the player to alter his or her math goals and open up new avenues of exploration in game play.

Finally, the prior understandings (Parameter 4) that children bring to the activity play fundamental roles in children's construction of mathematical goals in play. For instance, children who have difficulty understanding the denominational structure of blocks, may limit themselves to goals in which they merely conceptualize the problem as offering an overpayment, or they may seek assistance when accomplishing gold problems, relying on their partners to help them structure the sequence of goals that would lead to an adequate exchange.

The application of the four-parameter model to Treasure Hunt is providing analytic direction to the analysis of the interplay between social and cognitive developmental processes. We expect to find that children's goals can be understood as emerging with reference to each of the principal parameters in the course of problem formation and problem solution in play. For instance, we expect (a) younger children to form less complex arithmetical goals, (b) artifacts like doubloons to give rise to particular kinds of arithmetical goals, and (c) the social interactions between same- and mixed-ability groups to support differently the emergence of particular kinds of goals. The analysis of emergent goals also provides a basis to proceed with the second analytic component concerned with cognitive development: With knowledge of the arithmetical goals that children are engaged in the course of play, we can ask questions about the nature of the cognitive developments children are constructing to accomplish those goals.

Analyses of Children's Developing Understandings Linked to Their Emergent Goals (Component 2)

We are engaged with various analyses of developmental shifts in children's base-10 block-linked representations and strategies. First, we analyze shifts in both children's representations and strategies revealed in play as a function of session (first vs. last session of play), ability grouping, and grade level. For instance, with regard to representation, we find that children differed in how they translate block values in gold into the standard orthography. Some merely translated each denomination of block into a digit representation, and they frequently did not use the zero as a place holder [e.g., 8(100s), 5(1s) is represented as 85; 7(100s), 14(10s), and 11(1s) is represented as 71,411]; others produced more conventional translations. With regard to strategies, children differed in how they accomplished gold subtraction problems that involved trading...
(e.g., 1,006 – 18). One child may have engaged in a sequence of three trades: first trading the 1,000 block for ten 100 blocks [1(1,000s) → 10(100s)] and, in turn, trading one 100 block for ten 10s [1(100s) → 10(10s)], one ten for 10 1s [1(10s) → 10(1s)], and then making the payment of one 10 and eight 1s [1(10s) 8(1s)]. In contrast, another child may have abbreviated this transparent, but cumbersome strategy by merely trading the 1,000 block for nine 100s, eight 10s, and two 1s; the trades being implicit in the solution.

Second, we posttested children after play, using tasks designed to assess the kind of knowledge we suspected children would generate in attempting to accomplish the goals that emerged in play. These assessments included the representation of large numerical values, the mapping between the representations between base-10 materials and the standard number orthography, and arithmetical problem solving using base-10 materials. We administered these tasks to both the experimental groups and the control group. We compare the game players with the control group to determine whether and/or in what way the play of Treasure Hunt may lead children to mathematical understandings.

The Interplay Between Knowledge Generated Across Activities (Component 3): The Treasure Chest and the Gold Register

Under the prior Component 2, we are engaged in analyses of cognitive developments linked to children's efforts to structure and accomplish the emergent goals of play. These analyses include children's ability to represent numerical values with base-10 blocks and their ability to represent values using the standard orthography.

Component 3 is concerned with the interplay between knowledge children structure in one activity and their appropriation and use of that knowledge to accomplish emergent goals in another. For instance, we can ask, do children draw on one kind of representational system (knowledge of base-10 block forms, knowledge of the standard orthography) to help them understand the other? The following example shows that, on occasion, children make use of their understanding of base-10 block representations of quantity to produce representations in the standard orthography, and, at the same time, children make use of their understanding of the standard orthography to make sense of base-10 block representations. Further, the example shows the protracted processes often characteristic of the interplay between learning across activities in practices and the way such learning is interwoven with social
interactions. The example begins as one child, T, attempts to represent “one thousand and thirty-one” in his gold register using the standard orthography:

T’s treasure chest contains ten 100s pieces, three 10s pieces, and one 1s piece (see Fig. 15.8a); as he begins to change his gold register, he states incorrectly, “I have 100 and something” (apparently conceptualizing his 100s pieces as 10s pieces). T then counts his gold and puts “131” in his register (apparently treating the 100s piece as having a value of 10). As he places the last digit, T looks at the value that R reports in his register, 871, and realizes that he must have considerably more. . . . T exclaims, referring to his own ten 100s pieces, “Hey, that’s a thousand, boy.” Looking doubtful at T, R shakes his head “No.” T asserts, “Yes it is, yes it is!” and then recounts the 100s pieces, showing that there are 10 of them, and sticks out his tongue at R. Then both T and R recount the 100s pieces together, confirming that there are, in fact, 10 (see Fig. 15.8b). T then asks, “How do you make it 1,000 (referring to the “131” in his gold register)?” In an effort to solve T’s problem, R shifts T’s gold register from “131” to “131,” and both consider the new arrangement. T laments (only half seriously), “We don’t have any commas.” R then, with some excitement, incorrectly notes, “Oh, I know, you don’t have 1s” (R sees that he is wrong, shifting to the assertion), “You don’t have none of these (pointing to the empty 1,000s column in T’s treasure chest).” This insight does not lead R to a solution, and he looks frustrated, returning the digits to the initial 131 spacing. T then breaks in, “How do we get those (referring to the 1,000s column in his treasure chest, apparently playing off of R’s observation of the empty 1,000s column)?” Silence. Then T continues, but with sudden insight, “Oh, I know, I could take all these (T’s ten 100s),” indicating he could trade the ten 100s for a 1,000 block; T makes the trade (see Fig. 15.8c). T then looks at gold and exclaims, “one thousand and thirty-one!” At the same time, R is moving digits on the gold register so that they are arranged as 1 31 again. R says, “You got a zero. Put a zero right here (pointing to the blank space).” T looks at first as if he believes R is wrong. R prods, “Go ahead put a zero! That’s all, put a zero.” T follows R’s recommendation, placing the zero, saying (Fig. 15.8d) with clear reflective insight, “Oh yeah, it could have been ten hundred and thirty-one,” referring back to the original noncanonical arrangement of gold doubloons (ten 100s, three 10s, and one 1s).

In his efforts to determine and represent the value of his gold, T finds himself stumped. Two points are noteworthy in the ensuing solution: one about transfer and the other about social processes interwoven with transfer. With regard to transfer, rather than an immediate insight or generalization of one symbolic form to the other, transfer was a protracted process in which T was structuring and restructuring a solution across
FIG. 15.8a. As T begins to change his gold register, his treasure chest contains ten 100s pieces, three 10s pieces, and one 1s piece.

<table>
<thead>
<tr>
<th>T’s Treasure Chest</th>
<th>T’s Gold Register</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000s</td>
<td>100s</td>
</tr>
<tr>
<td></td>
<td>10s</td>
</tr>
<tr>
<td></td>
<td>1s</td>
</tr>
</tbody>
</table>

FIG. 15.8b. After counting his gold, T places the numerals, “131” in his gold register.

<table>
<thead>
<tr>
<th>T’s Treasure Chest</th>
<th>T’s Gold Register</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000s</td>
<td>100s</td>
</tr>
<tr>
<td></td>
<td>10s</td>
</tr>
<tr>
<td></td>
<td>1s</td>
</tr>
<tr>
<td></td>
<td>131</td>
</tr>
</tbody>
</table>

FIG. 15.8c. Prompted by R's observation of the empty thousands column in T’s treasure chest, T excitedly trades ten 100s pieces for one 1000s piece.

<table>
<thead>
<tr>
<th>T’s Treasure Chest</th>
<th>T’s Gold Register</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000s</td>
<td>100s</td>
</tr>
<tr>
<td></td>
<td>10s</td>
</tr>
<tr>
<td></td>
<td>1s</td>
</tr>
<tr>
<td></td>
<td>131</td>
</tr>
</tbody>
</table>

FIG. 15.8d. T places the “0” appropriately to represent 1.031.

<table>
<thead>
<tr>
<th>T’s Treasure Chest</th>
<th>T’s Gold Register</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000s</td>
<td>100s</td>
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<td></td>
<td>10s</td>
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<tr>
<td></td>
<td>1s</td>
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<td></td>
<td>1031</td>
</tr>
</tbody>
</table>
the two partially isomorphic symbolic forms—the number orthography and the gold doubloons. The solution path was one in which the child first struggled with the value of the doubloons (T first asserting "I have 100 and something...", but later, "Hey, that's a thousand..."). However, T's accurate determination of the doubloon value is only an early intermediate step in the construction of an appropriate orthographic representation. T knows that his orthographic representation of "131" is not an adequate reflection of the value of "one thousand and thirty-one," and he is puzzled by the mapping across that orthographic and doubloon forms. With his realization that he can obtain a 1,000 doubloon by trading ten 100s, there is progress in achieving a doubloon configuration in which the mapping across forms is more transparent; however, there is no immediate insight. Finally, with R's directive to insert the zero, the mapping from doubloons to the orthography is realized, and with T's assertion, that "it could have been ten hundred and thirty-one," the mapping from the orthography to the original doubloon configuration is realized. Clearly, transfer as it occurs in this example is an extended process of construction, one in which processes of problem representation in both symbolic forms and mapping across forms are interwoven with one another in a microgenetic process.

With regard to social interactional processes in the transfer, we can ask, did T or R accomplish the transfer? Although certain elements of problem solution were attributable to T or R, respectively, when we consider the process as a whole, each child's contribution was dependent on the contribution of the other. It was R who first noted that T had no 1,000s pieces, and this, in turn, prompted T to effect a trade of ten 100s for one 1,000s piece. T's new doubloon configuration, in turn, led R to the insight of the missing zero. The subsequent change in the zero led T to realize that "1,031" could be understood as "ten hundred and thirty-one" as well as "one thousand and thirty-one." Clearly, the interplay between knowledge of the two symbolic forms involved a joint social process in which the emergent goal was accomplished through children's orchestrated contributions.

Our analyses of the interplay between learning across activities and practices extends beyond case studies. We suspect that the mathematics that the children generated through the play of Treasure Hunt was useful to their efforts to make sense of the algorithmic-based math in the classroom. To test this thesis, we are conducting two analyses.

First, included in the posttest for Treasure Hunt (Phase 1) were some arithmetical problems in standard format for school. We are contrasting the game players' solutions to these problems with matched controls who had not had game-playing experience. We expect to find that Treasure Hunt players...
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Hunt players make use of their manipulative-based mathematics in their solutions to paper-and-pencil computational problems.

Second, we are analyzing our Phase 2 data to determine whether game players learned more from our classroom lesson than the nongame players. We expect some indication that game play is useful to children in structuring an understanding of the algorithmic-based arithmetic that children are presented in school.

CONCLUDING REMARK

In this chapter, I described a research framework that was designed to reveal the interplay between social and cultural processes in individuals’ developing practice-linked mathematical understandings. Research generated by the framework informed the development of a classroom practice that supports children’s sustained engagement and children’s mathematical sense making. Preliminary analyses of Treasure Hunt suggest that it has promise as a useful educational activity that supports children’s construction of rich mathematical environments. These same preliminary analyses of Treasure Hunt also show the promise of the framework for gaining further insight into children’s development of mathematical understandings in practices.

ACKNOWLEDGMENTS

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