Chapter 13

A methodological framework and empirical techniques for studying the travel of ideas in classroom communities

Geoffrey B. Saxe, Maryl Gearhart, Meghan Shaughnessy, Darrell Earnest, Sarah Cremer, Yasmin Sitabkhan, Linda Platas and Adena Young

Powerful methods for analyzing the travel of ideas are essential for understanding the process of learning in classroom communities. In this chapter, we argue that a genetic perspective provides a useful methodological frame. We describe specific empirical techniques for the study of the travel of ideas, and illustrate our approach with findings from recent investigations of lessons on integers and fractions for fifth grade.

We begin with a summary description of a classroom exchange in a classroom that supports inquiry-oriented mathematics instruction.

Students in Ms. Jones’s classroom begin their math lesson by working on the problem: How many equivalent fractions can you name for Point A on the number line? After a brief class discussion of the range of answers, Ms. Jones sends students to small groups to explain their thinking and try to reach consensus on one solution. In one group, students argue about different ways of re-partitioning the number line to generate names of equivalent fractions. In another group, students share ways of multiplying the fraction by one (2/2, 3/3, etc.) to generate equivalent fractions. As Ms. Jones observes groups at work, she notices that some students are changing their thinking, while others persist with their original solutions. Ms. Jones re-convenes the class and orchestrates a discussion, building upon students’ ideas to guide students toward richer interpretations of equivalence, as she sets the stage for a future lesson on estimating and comparing points on a number line.

A guiding principle of inquiry-oriented mathematics instruction is that student thinking is a resource for teaching and learning. Teachers pose problems, solicit students’ solutions and orchestrate discussions to guide students to make their mathematical ideas public and reflect on relationships among their ideas. In the social context of classroom lessons, students’ ideas may be taken up or rejected, valued or devalued, and interpreted in various ways. And, in this process, students’ mathematical ideas become elaborated and often transformed. In this chapter, we argue that investigating the emergence, travel and transformation of ideas is essential to understanding learning and teaching in inquiry-oriented classroom communities.

Our chapter introduces a methodological framework and specific empirical techniques for the study of the travel of ideas, and we illustrate our approach
with findings from recent investigations of lessons on integers and fractions for fifth grade. We begin with the principles that guide our design of lessons that support whole-class and small-group discussions. Next, we present our framework for investigating the travel of mathematical ideas during the lessons, with a particular focus on the constructs we are using to understand processes of change: (a) students' learning over the course of a lesson through short-term ontogenetic processes; (b) students' moment-to-moment construction of representations and communications through microgenetic processes; and (c) students' propagation of ideas in local interactions in classroom communities through sociogenetic processes. The third section illustrates our approach with an analysis of the travel of ideas in a single classroom lesson. We conclude with reflections on our methods and the challenges and prospects for future work.

**Design and investigation of inquiry lessons on integers and fractions**

Integers and fractions are challenging topics, and our goal is to design a curriculum unit that engages diverse students with the mathematics in sustained and coherent ways. Our approach to lesson design builds on socio-constructivist assumptions about meaningful learning: In the context of mathematical activities where students work with others and independently, students develop understanding as they produce, coordinate and adapt representations (number lines, area models, Hindu–Arabic arithmetical procedures) to serve mathematical functions (subtract negative numbers, compare fractions) in communicative and problem-solving activities. Lesson sequences should support students' efforts to (a) extend their own sometimes idiosyncratic forms of mathematical representation for integers and fractions to serve new and important mathematical functions, and (b) incorporate new forms of representation valued in school to serve and extend mathematical functions that they already know. Thus we view learning as shifts in the relationships that students construct between mathematical forms and mathematical functions.

Our principles for lesson design are grounded in our socio-constructivist assumptions about the ways that mathematical ideas travel and are transformed.

1. **Lessons should target core mathematical ideas.** Elementary mathematics textbooks in the United States contain too many mathematical topics and representations to be covered in depth, and students cannot be expected to develop rich connections within and across topics and representational forms. We are focusing on core generative ideas and setting aside more peripheral topics and representational forms. For fractions, we consider equivalence to be one core idea and the number line one core representational form. Equivalence is the basis for the principle that any particular representation of a rational number is not the number itself, and any rational number can be represented in an infinite number of ways. As a representational form, the number line can support elementary students' understanding of equivalence, order and magnitude of rational numbers; it then becomes a critical tool for secondary school students' later work with the Cartesian coordinate system and mathematical functions.

2. **Lessons should engage all students.** US classrooms are diverse, and we need
pedagogies that support the intellectual engagement of students who vary in mathematical understandings, interests and investment. This principle led us to the six-phase inquiry lesson structure depicted in Figure 13.1, a structure that encourages active participation, reflection and questioning about mathematics (Saxe et al. 2007). The lesson is organized around a Problem of the Day that is challenging and yet accessible — students must choose one answer among several alternatives that represent common patterns of student thinking. The multiple-choice format for the Problem of the Day is an adaptation of the Itakura method originally developed for science lessons (cf. Inagaki, Hatano & Morita 1998). To support student exploration of form–function relationships between fractions, representations and the functions they can serve, students revisit the Problem several times over the course of the lesson through individual work, small-group interactions, whole-class presentations and teacher-led discussions.

Phase 1: Independent work and pre-assessment. Students work independently to solve a problem like the one illustrated in Figure 13.1, ‘How many fraction names for point A?’ To provide a scaffold for student engagement, students choose one of five multiple-choice alternatives and then justify their choice in writing. The answer choices are based on previous studies of students’ reasoning on similar problems (Saxe, Langer-Osuna & Taylor 2006; Saxe, Taylor, McIntosh & Gearhart 2005): (a) only one fraction name; (b) two fraction names; (c) between 3 and 10 fraction names; (d) between 11 and 20 fraction names; (e) more than 20 fraction names. Students’ answers are their initial forays into the mathematics as well as the teacher’s initial assessment of the range of ideas in the classroom. The teacher then chooses several students who represent that range to make presentations in Phase 2.

Phase 2: Student whole-class presentations. The teacher invites several students to present their solutions. During the presentations, other students hear the solu-

![Figure 13.1](image_url)  
*Figure 13.1* An illustration of the six-phase inquiry lesson structure.
tions they themselves had chosen as well as solutions they had considered and rejected, along with their peers’ justifications.

**Phase 3: Small-group discussions.** Students meet in small groups to present their thinking to one another and try to reach consensus on a single solution and justification. The requirement that each student presents as well as listens supports student engagement.

**Phase 4: Opportunity for revision.** Students revisit their previous solutions to the Problem of the Day. They are invited to explain how their revisions – or their decisions not to revise – have shifted based on the whole-class presentations and small-group discussions.

**Phase 5: Teacher-orchestrated discussion.** Drawing on students’ ideas in student presentations and small-group discussions, the teacher guides students through an exploration of contradictions. The discussion concludes with extended discussion of the correct solution.

**Phase 6: Independent work and post-assessment.** The class concludes with independent work on two extension problems that parallel the Problem of the Day.

3. **Lessons should be organized as a mathematically coherent series.** What students learn in one lesson should build on prior lessons and support learning in subsequent lessons. In our lesson on identifying a point on a number line with unequal partitions, for example, the answer ’1⁄4’ (Figure 13.2) was designed to foreshadow the next lesson on relationships between equivalent names such as 2/8 and ¼ for the same point on a number line. Students’ work with the non-routine number line with unequal partitions (Figure 13.2) was designed to raise issues about the necessity of equal intervals that students will address again when reflecting on relationships between 1⁄4 and 2/8.

**‘Travel of ideas’ in the classroom**

In our approach to analyzing the travel of mathematical ideas in and across lessons, we distinguish between a methodological approach and empirical techniques. Our methodological approach begins with two framing questions – the what? and the how? of travel.

![Problem of the day: Figure out what this point is called](image)

Circle correct answer and justify:

- 2/6, 2/7, 1/4, 2, 2/4

*Figure 13.2 Problem of the Day for the lesson preceding the equivalent fractions lesson.*
What are the ‘ideas’ that travel?

The ontology of mathematical ideas is a topic of polemical discussions across epistemological and psychological frameworks. Our account builds upon the work of theorists who locate ideas in the constructive activities of individuals (Piaget 1970; Sfard 2008; Vygotsky 1986): We treat mathematical ideas as emerging and becoming crystallized as students participate in the collective practices of classroom life. The process occurs as students construct connections between representational forms and the mathematical functions that they use those forms to serve in communicative and problem-solving activities (Saxe 1994; Saxe & Esmonde 2005). In the equivalent fractions lesson, the ideas that travel are students’ shifting uses of geometrical and arithmetical forms as students conceptualize, discuss and solve problems about fractions.

Consider, for example, the ideas that the lesson supports as students explore ways of answering a Problem of the Day on equivalence. In Figure 13.3, the left column illustrates connections that students might generate between ‘two-thirds’ and equivalent fractions by constructing geometric forms on the number line (e.g., hatch marks, interval lengths); as students explore geometric operations on the line, some may eventually conjecture that it is possible to produce an unlimited number of divisions of the line. The right column of Figure 13.3 illustrates connections students may generate between ‘two-thirds’ and equivalent fractions through an arithmetic form, multiplication of \(\frac{2}{3}\) by the fractional equivalents of 1; as students explore arithmetic transformations of equivalence, they may come to understand that the series of multiplications can proceed indefinitely. Students may also explore relationships between geometric and arithmetic forms — for example, they may observe that every geometric repartitioning of the line corresponds to an arithmetic multiplication of the fraction by the equivalent of 1. Partitioning each distance that is equivalent to \(\frac{1}{3}\) of a unit into halves to create

\[
\begin{align*}
\text{Geometric transformations} & & \text{Arithmetic transformations} \\
\text{‘four-sixths’} & & \frac{2}{3} \times \frac{2}{2} = \frac{4}{6} \\
\text{‘six-ninths’} & & \frac{2}{3} \times \frac{3}{3} = \frac{6}{9} \\
\text{‘eight-twelfths’} & & \frac{2}{3} \times \frac{4}{4} = \frac{8}{12} \\
\text{‘ten-fifteenths’} & & \frac{2}{3} \times \frac{5}{5} = \frac{10}{15}
\end{align*}
\]

Figure 13.3 Geometric and arithmetic forms used to explore equivalent names for the same point on the number line.
sixths corresponds to the multiplication of 2/3 by two halves (2/2), or 2/3 \times 2/2 = 4/6; similarly, repartitioning each third into thirds corresponds to the multiplication of 2/3 by three-thirds (3/3), making ninths, or 2/3 \times 3/3 = 6/9, and so forth.

**How do ideas travel?**

The collective practices of the classroom support the emergence, reproduction and alteration of mathematical ideas in students’ problem solving and discussion. All participants contribute to the collective practices of the lesson – to the lesson’s emerging structure, to the use of valued forms of representation and associated functions, and to the social positions of students and teacher (e.g., how students value, devalue or ignore one another’s actions and contributions). The activities of individuals are in part constitutive of this structure but also take form in relation to it, and it is in the dynamic between individual and collective activity that we locate ‘travel of ideas.’ We analyze the travel and transformation of ideas in the activities of individuals through three coordinated strands of genetic analysis: ontogenesis, microgenesis and sociogenesis. Though these strands are integrated in the nexus of activity, their analytic separation is critical to an understanding of learning and development. **Ontogenesis** focuses on shifts in patterns of thinking over the development of the individual; **microgenesis** involves the construction of meaningful representations in activity; and **sociogenesis** entails the reproduction and alteration of representational forms that enable communication among participants in a community. For each strand, we assemble somewhat different but overlapping sets of evidence. As we will show in the section on empirical techniques, our methodological approach is a framework for producing evidentiary claims about the genesis of ideas, rather than a fixed set of distinct procedures for collection and analysis of data for each strand.

**Ontogenesis.** Individual development of mathematical thinking – its ontogenesis – is marked both by continuity in the individual’s ways of understanding the experienced world and discontinuity as the individual structures new systems of understanding out of prior ones. As Langer (1970: 733) has expressed it, ‘The central theoretical issue is dialectical: How does a developing organism change qualitatively and at the same time preserve its integrity?’ The answer to this question was the life’s work of Piaget (1970), Vygotsky (1978), and Werner (Werner 1948; Werner & Kaplan 1962), and our interest in understanding developmental relations between forms and functions builds upon their work when we examine qualitative shifts in student thinking over time. But unlike most treatments of ontogenetic change where the focus is on major time spans of intellectual development (e.g., infancy to early childhood, middle childhood to adolescence), we focus on continuities and discontinuities in students’ thinking over the course of a single lesson. Are students creating new functions for forms that they have used previously? Are they incorporating new forms linked to classroom life for functions that they already have understandings? If so, how could the character of such changes be documented and analyzed?

To date, our investigations of the ontogenesis of ideas have focused on shifts
in individual students’ uses of mathematical forms for particular functions from the beginning to the end of each lesson. As our research proceeds, we will also track shifts in form–function relationships over the course of lesson sequences.

Microgenesis. Microgenesis is the process of moment-to-moment construction of representations as individuals work to turn representational forms into means to serve mathematical functions. In the classroom, representational forms like number lines or fraction words contain no inherent mathematical functions, and mathematical functions like solving rational number or whole number problems can be served by any number of representational forms. Relationships between forms and functions emerge and shift in students’ moment-by-moment activities as students appropriate and adapt forms to accomplish local and emerging goals (Saxe 1991). We illustrate some properties of microgenetic processes with examples of students’ efforts to identify a fraction on the number line and produce equivalent names for the fraction.

A child working on the Problem of the Day in Figure 13.1 may be coordinating multiple forms and potential mathematical functions to identify a name for Point A. If she conceptualizes the task in terms of fractions, she needs to: conceptualize the geometrical form of the number line as partitionable into three intervals that are three equivalent fractional parts of a whole; conceptualize a register of fraction word forms as a sequence (‘zero, one-third, two-thirds’) with each successive term representing a magnitude equivalent to the prior term in the sequence; construct correspondences between her conceptualization of the partitioned line and her conceptualization of the lexical forms so that ‘two-thirds’ comes to refer to the endpoint of the second interval (Point A). Each of these conceptualizations is challenging. Students’ idiosyncratic conceptualizations and coordinations will vary, and their understandings will shift in different ways as they participate in and contribute to independent and collective activities over the course of the lesson.

Let us consider two hypothetical students’ use of geometric forms to identify additional names for Point A in Figure 13.1. One student inserts a hatch mark between 1/3 and 2/3, counts the hatch marks on the number line in Figure 13.1 starting with 0, and decides that Point A is ‘three-fourths’ because it is the third hatch mark of four (see Figure 13.4a). This is one microgenetic construction; shortly after, when he notices that some other students are adding additional hatch marks to their number lines, he inserts a second hatch mark to his line, and then decides that another name for Point A can be ‘four-fifths.’ In these two successive microgenetic constructions, this student has revised the geometric form of his representation, though he has used his new representation for the same mathematical function (assigning values to points based on counting hatch marks). Another student, also using a geometrical form, conceptualizes fractions on the line as equally partitioned intervals, and labels Point A initially as ‘two thirds.’ When asked to produce another name for Point A, she adds three new hatch marks at the midpoints between the existing hatch marks (Figure 13.4b) and uses them as a means to identify the equivalent name of ‘four-sixths’ (Figure 13.4b), again coordinating lexical with geometrical forms in the act of quantification. When a tablemate suggests to her that there are infinitely many names for Point A, she resists, arguing ‘you can’t fit in that many marks on the line.’ Thus, while she extended her conception of equal partitions to produce ‘four sixths,’
she does not appreciate that the physical form can be treated as a representation of the idea that the partitioning process can be repeated indefinitely.

Both students used hatch marks as geometric means to identify a new name for the point, but their uses of the hatch marks served different functions — to add one additional mark and recount (Figure 13.4a), or, to partition the line between 0 and 1 into equal intervals and re-compute the relationship between parts and whole (Figure 13.4b). Nonetheless, in both cases, these children have generated new mathematical ideas in their microgenetic constructions.

Our investigations of the processes of microgenesis focus on students’ uses of mathematical forms for particular functions. We select cases for analysis based on findings from ontogenetic analyses of change in form–function relations as well as sociogenetic analyses (next) of the case students’ roles in communication, uptake and alteration of mathematical ideas.

Sociogenesis. Sociogenesis is the reproduction and alteration of ideas over time in the classroom community. When students are engaged in discussion or working jointly on a problem, they express their ideas in particular ways to help their listeners understand and appreciate what they are saying. Mathematical talk serves both personal and interpersonal functions, and, at times, what students present to others may not fully represent their understanding. At the same time, when students listen to others, the sense they make of what others are saying or writing may not be fully in accord with the speakers’ intentions. Communicative and collaborative acts have unintended consequences, and, over time, classroom communities unwittingly sustain or alter ways of talking about mathematical topics and solving mathematical problems as they make efforts to communicate with representational forms (Croft 2000; Evans 2003; Keller 1994; Saxe & Esmonde 2005).

Investigation of sociogenetic propagation of ideas requires analyses of the social processes that shape communication and uptake of representational forms and the functions they serve. We use a suite of methods to document students’ perceptions of other students’ influence on their mathematical ideas, as well as patterns of communication, uptake and alteration of ideas over the course of the lesson.

Onto-, micro- and sociogenetic processes are intrinsically related in the travel of ideas. In the context of whole-class or small-group discussions, students may use the products of one another’s microgenetic constructions, and in so doing, unwittingly participate in the reproduction and alteration of forms and functions in processes of sociogenesis. At the same time, the production and uptake of

![Figure 13.4](image)

*Figure 13.4 The products of two microgenetic constructions of equivalent names for Point A (the dotted hatch marks were added by students).*
Methodological framework and empirical techniques

Emerging mathematical ideas are enabled and constrained by the ontogenesis of representational activity. We situate our analyses of the travel of ideas in the classroom in the interplay of genetic processes as they emerge in the collective practices of students and teachers.

Empirical techniques

Our methods for investigating the travel of ideas in classroom communities are eclectic. We collect data from a wide range of sources, and use diverse methods of data reduction and integration to reveal onto-, micro- and sociogenetic processes. Figure 13.5 is a sketch of the data sources for the equivalent fractions lesson. Student worksheets and interviews provide our primary evidence for analysis of ontogenetic change in relations between forms and functions. Video of whole-class and small-group interactions as well as students’ reports of the classmates who influenced their thinking provide primary evidence of sociogenetic processes. Case analyses of microgenetic processes draw from all data sources.

Ontogenetic analyses: shifts in form–function relations in student thinking over the course of the lesson

We coded students’ worksheets to document shifts in the forms of representation that students used and the functions that they used the forms to serve in solving the Problem of the Day. Figure 13.6 is a summary of group results for the first and last worksheets (left and right panel, respectively). In each panel, the bars represent the forms students used in their solutions: geometric (e.g., using hatch marks to repartition the number line); arithmetic (e.g., multiplying a number by the fractional equivalent of 1/2, 3/3, etc.) to create new names for Point A; other (such as words like ‘percent’ or ‘decimal’ that describe the type of representation). The partitions within each bar show the four functions that the forms served. Non-normative solutions could not lead to equivalent fraction names for Point A; partial solutions were incomplete procedures to generate equivalent fraction names; procedural solutions were appropriate steps to generate new fraction names but without an explanation of the logic of the procedure; principled consisted of a correct procedure as well as an explanation of its logic. These group results provide an initial description of cohort shifts in mathematical thinking, and set the context for coordinated analyses of microgenetic shifts in individual students’ understandings of relationships between forms and functions (as we explain in a subsequent section).

On the first worksheet, the forms used by most students (17 of a class of 26) were coded as ‘other,’ because the forms were neither arithmetic nor geometric. A minority of students (9 of 26) used geometric forms (partitioning the line further) or arithmetic forms (multiplying the fraction identified on the line by another number), but only one of these students used these forms to serve principled functions. These group results from the first worksheet indicated considerable room for progress in the classroom. By the end of the lesson (Figure 13.6, right panel), the class shifted in the ways they accomplished the equivalence problem, both in the forms used and the functions that they used those forms to
Figure 13.5 Data sources for studying the travel of ideas.
serve. Of 26 students, 20 used arithmetical and/or geometrical approaches, and, of those students, two used them in principled ways to serve normative functions, and six showed knowledge of procedures to accomplish the problem.

Group trends provide a summary portrait of ontogenetic change, while case analyses enable us to examine continuities and discontinuities in students’ thinking over successive data points in the lesson. Each time that Damian, for example, solved an equivalence problem, the mathematical forms he used and the functions he used those forms to serve shifted.

Damian’s initial conception of the problem on worksheet #1 was that there were two ways to name an equivalent fraction, ‘decimal or percent.’ That conception did not involve any approach to computation of equivalent fractions, and we coded
normative.' After a normative defining that there are explained that he line (a geometric 
3 'one-eighth,' and ne-eighth by one-
sent fraction. For 1 arithmetic forms (sic numerals) for action was unclear the final debrief, 
dural,' because he int the names of (instead was to identify instead of 2/3). For equivalent fraction an arithmetic pro-
sumption of the Day on it, '1/3,' '2/3,' and equal intervals, and

s case analyses, we veen mathematical ters on equivalence develop techniques to

were more than 20 1's efforts to identify. His initial approach arithmetic multipli-
ing. In the geometric, the points marked own unit, and then- to count ha replacing the intervals, etc., thereby

-structuring lexical word forms and the intervals generated on the line into quantitative meanings in relation to one another. In the arithmetic and final part of his solution, Damian multiplied 1/3 by 1/8, and offered 'one twenty-fourth' as an example of one equivalent fraction for the labeled point (2/3). Here Damian was coordinating arithmetic procedures with lexical forms to produce a solution which was arithmetically correct, but unrelated to either the initial geometric number line representation or his transformation of it. Thus, as Damian was solving the class Problem of the Day, he was constructing goals that were his own, and his microgenetic constructions of relationships between forms and functions emerged in relation to his goals.

Students in Damian's class varied markedly in the ways that they made sense of the equivalence problem. While each student was using forms as means to construct solutions to the problem, their solutions were the products of their own microgenetic constructions. Consider the ways that three other students used lexical, geometric and/or arithmetic forms as means to solve the Problem of the Day on the first worksheet.

Figure 13.7 contains the overhead transparencies that the students created as they explained their solutions to the class. Sienna (left panel) used an arithmetic procedure, and she stated that there are infinitely many names, because you can keep multiplying the fraction by 2.' Daniel (middle panel) used a geometric form (hatch marks) to explain how to find one new name for the fraction 2/3; he partitioned each interval on the given line into half, determined that Point A could also be called four-sixths in addition to two-thirds, and claimed that there were no other names for Point A. Anabelle (right panel) used both arithmetic and geometric forms. She explained that one could either continue adding hatch marks or multiply both the numerator and denominator by 2, adding that doubling the number of hatch marks is the same as doubling the (numerator and denominator of the) fraction.

Sienna's arithmetic rule to multiply a fraction by 2 is – if interpreted literally – incorrect, because it cannot lead to the production of an equivalent fraction. But Sienna's actions and her verbalizations were not well-aligned, leaving us (and,

Figure 13.7 Ideas articulated by Sienna, Daniel and Anabelle in whole-class presentations (Phase 2).
the forms that he used as 'other' and the functions as 'non-normative.' After whole-class and small-group discussions, Damian shifted toward a normative definition of equivalent fractions in his second worksheet, responding that there are 'more than 20 names for point A.' In a debriefing interview, he explained that he added seven hatch marks between 0 and 1/3 on the number line (a geometric approach), named each of the new intervals between 0 and 1/3 'one-eighth,' and then incorporated an arithmetic approach by multiplying one-eighth by one-third, yielding 'one twenty-fourth' as one example of an equivalent fraction. For this solution, we coded Damian as using both geometric and arithmetic forms (hatch marks and arithmetic multiplication with Hindu–Arabic numerals) for non-normative functions (using a procedure for which the function was unclear in relation to the problem). On the final worksheet and in the final debrief, Damian's reasoning about equivalence was coded as 'procedural,' because he used computational procedures to generate correct equivalent fraction names without explication of principles. The task on the final worksheet was to identify how many names there were for the marked point of 2/5 (instead of 2/3). For the first time, Damian correctly identified the point and four equivalent fraction names; although he provided no explanation, his names were an arithmetic progression (2/5, 4/10, 8/20, 16/40) suggesting the use of an arithmetic procedure. In the debrief interview, Damian re-solved the initial Problem of the Day on worksheet #1 by writing on the number line from left to right, '1/3,' '2/3,' and '3/3,' repartitioning the number line into six approximately equal intervals, and re-labeling the hatch marks as sixths.

Through pre-post comparisons of group results as well as case analyses, we document ways that students reconstruct relationships between mathematical forms and functions as they work independently and with others on equivalence problems. As we develop sequences of lessons, we will also develop techniques to follow ontogenetic progress over longer trajectories.

**Microgenetic analyses: turning forms into means to serve mathematical functions**

Each data point that we used in analyzing Damian's ontogenetic progression over the course of a lesson constitutes a microgenetic construction, an occasion when Damian turned forms like number lines, fraction words, arithmetic procedures into particular meanings as he used them to cognize and accomplish goals in activity. Damian's effort on his second worksheet is a useful illustration.

Damian's answer on the second worksheet was that 'there were more than 20 fractions' for a point (2/3) on the given number line. Damian's efforts to identify equivalent fractions for 2/3 were complex and protracted: His initial approach was through geometric partitioning, and he then applied an arithmetic multiplication to (his interpretation of) the results of his partitioning. In the geometric part, Damian began by inserting seven hatch marks between the points marked 0 and 1/3 on the number line, inadvertently creating his own unit, and then calling each of the resulting intervals 'one-eighth.' He appeared to count hatch marks and use the product of his count as the basis for naming the intervals 'eighth' and the sequence of points as 'one-eighth,' two-eighths,' etc., thereby
students were structured in their right varied resources will show next, their e resources for other consensus on the correct

\section*{Conclusions in the class, and we investi- alter how they representreintroduced and their study of sociogenesis y propagate through

, we drew upon multimates for math work ts' initial social posihole-class discussions ts' microgenetic con reports of who influen and post-lesson cial influences within

relation to students' resents relationships fluence on students' he/she wanted to sit 'choosers,' the right students' choices. As are chosen most freuin and Pamela, sughiss class as resources who reported being after the lesson. The

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{sociogram.png}
\caption{Students' preferred tablemates on the initial sociogram in relation to social influences reported on the second worksheet and final debrief.}
\end{figure}

noteworthy finding here is that the three students who were cited most frequently – Sienna, Anabelle and Daniel – were the three students who had made wholeclass presentations (Figure 13.7), and none of these students was widely perceived before the lesson as valued tablemates. The discrepancy between preferred tablemates and self-reported influence suggests that students' presentations can have striking influence on the travel of mathematical ideas even when the presenters are not widely viewed as valuable members of small groups in math class. The value of these students to their classmates emerged within the context of the
most likely, Sienna) somewhat uncertain of her understanding; in action, she appropriately doubled both the numerator and denominator and indicated that the act could be repeated again and again. In Daniel’s geometric construction, he showed limited understanding that the partitioning could be repeated indefinitely, appearing constrained by the geometric representation of the fraction on the line, a hurdle that did not occur with Sienna’s arithmetic approach. Annabelle cited both an arithmetic multiplication and a geometric partitioning as ways to show that there is an infinite number of equivalent fractions, though she did not articulate the relationship between the two approaches.

In each of their microgenetic constructions, these three students were structuring representational forms to accomplish goals that they constructed in their efforts to solve the Problem of the Day. These students brought varied resources to their work, and they constructed varied insights. As we will show next, their public presentations of their mathematical ideas then became resources for other students when they met in small groups to negotiate consensus on the correct answer.

Sociogenetic analyses: propagation of forms and functions in the classroom community

Sienna, Daniel and Annabelle presented their work to the class, and we investigated if and how the activity of presenting ideas led them to alter how they represented their solutions. How were the forms the presenters introduced and their mathematical functions taken up by other students? The study of sociogenesis focuses on the reproduction and alteration of ideas as they propagate through communications within a community.

To investigate the propagation of ideas in the classroom, we drew upon multiple data sources. Sociograms of students’ preferred tablemates for math work collected prior to the lesson provided evidence of students’ initial social positions. Videotapes of social interactions in small-group and whole-class discussions were our resource for analyses of the give and take of students’ microgenetic constructions from a sociogenetic perspective. Finally, students’ reports of who influenced their thinking during the lesson (second worksheet and post-lesson debrief) served as evidence of the social organization and social influences within the classroom.

Figure 13.8 contains results of the initial sociogram in relation to students’ reported influences on their thinking. The figure represents relationships between initial social positions and subsequent social influence on students’ mathematical ideas.

In the sociogram (left panel), each student chose whom he/she wanted to sit with during math class; the left column represents the ‘choosers,’ the right column represents the ‘chosen,’ and the connecting lines students’ choices. As indicated in the shaded cells, the three students who were chosen most frequently as desired tablemates for math work were Lyle, Jaquin and Pamela, suggesting that these three students had privileged status in this class as resources for mathematical ideas. The right panel connects students who reported being influenced and the students they identified as influential after the lesson. The
Methodological framework and empirical techniques

and after stu-

after the pres-

Craig were the 

how Craig 

ning to Craig 

are only two

; but yeah… 

was. At first I 

ths. Well that 

you can get

it there is more 

on Daniel’s 

generate more 

it or…. He 

it on your

still double

h, commenting 

ake it bigger 

her side as cap-

‘zooming in’ 

ving distance 

t works both 

well as multi-

ments on the 

take the dots,

broach] If you 

until you had,

ng expressed as 

train of equal 

ometric] But

geometric constructions in their abbreviated communications. It was often unclear 
what they were understanding at any given moment or what they intended to 
convey to one another. For example, what was Craig trying to convey in turn 5 
when asserting that ‘like all you have to do is make it bigger’ (the stretching 
gesture captured in Figure 13.9)? Was it the idea that the number line can be 
stretched, preserving the ratio between hatch marks to create room for more 
hatch marks? Craig’s talk and gesture may have been interpreted in any number 
of ways, and certainly the same can be said about Anabelle’s abbreviated refer-
ence to arithmetic procedures. From a sociogenetic perspective, what we see here 
is how ideas like Daniel’s may be devalued in interaction, and how ideas like Ana-
abelle’s may be brought forward. But the abbreviated nature of the exchanges, 
while perhaps interpretable to someone who already understands what is 
required for a solution to the Problem of the Day, may leave someone who is less 
formed uncertain about what is being asserted and why. Damian’s worksheet 
#2, which was completed immediately following this interaction, suggests that 
Damian appreciated the conclusion of Anabelle and Craig (more than 20 equiva-

tent fractions), but he took away the idea of partitioning and multiplication 
without rich understanding of ways they could be coordinated to solve the 
Problem of the Day.

To provide an index of the ways that students incorporated the presenters’ 
mathematical ideas, we examined shifts from the first to the last worksheet for 
those 14 students who cited Sienna as the most influential. (Sienna was the most 
frequently identified as influential, and 14 students was a reasonable sample for 
analysis.) Figure 13.10 contains the form–function distributions for each work-

![Figure 13.9 Small group composed of Damian (lower right), Craig (left), and Annabelle (upper right) during Phase 3 of the lesson, with Craig offering a conjecture with accompanying stretching gesture.](image-url)
lesson, after they explained their mathematical thinking to the class and after students had the opportunity to continue working on the problem in collaboration with other students.

To examine the role of the presenters’ ideas in the small groups after the presentations, we analyzed videotape of the group discussions. We illustrate our findings from one group shown in Figure 13.9. Damian, Anabelle and Craig were the participants, and you will see in the dialogue reproduced below how Craig rejected one of the presenter’s ideas (Daniel’s) and how Anabelle elaborated the ideas she had previously presented. In the excerpt, Damian is listening to Craig and Anabelle; Craig is reflecting on Daniel’s solution that there are only two names, ‘two thirds’ and ‘four sixths.’

CRAIG: [Politely critical of Daniel’s assertion] I realized Daniel was close, but yeah…
ANABELLE: [Interjects, explaining her own solution] Like, see what I did was. At first I got two-thirds. And I was like, well that’s… Then I got four-sixths. Well that would only be two names. And I was like, oh! If you times it by 2 you can get more.

CRAIG: [Appearing to make an association with Anabelle’s comments that there is more than one way (‘times it’ and partitioning), continues to reflect critically on Daniel’s incomplete geometric approach, pointing out that it could be used to generate more names] There are two ways. You either just keep doubling it or…. He [Daniel] doesn’t understand. It doesn’t matter if you can’t fit it on your paper. It’s still, like…
ANABELLE: [Picks up on part of Craig’s statement] Yeah. I know. You can still double it.

CRAIG: [Continues his critical reflection on Daniel’s geometrical approach, commenting on making the number line ‘bigger’) Like, all you have to do is make it bigger [gestures as if expanding the number line as he outstretches hands to either side as captured in Figure 13.9]. [In his debrief interview, Craig also talked about ‘zooming in’ on the number line, creating more space for hatch marks but conserving distance relations.]
ANABELLE: [Responds] Yeah, you can either double it or times it. It works both ways. [She is referring to ‘doubling’ (introduced by Craig in turn 3) as well as multiplying the numerator and denominator by the same value.]
CRAIG: [Still continuing his critique of Daniel’s geometric approach, comments on the effects of repeated partitioning of the line] You can actually just like make the dots, like…
ANABELLE: [Follows up on Craig’s ideas and takes up the partitioning approach] If you have this number line, you can just keep going like, bam, bam, until you had, like marks like this small.
CRAIG: [Attempts to interject a constraint on Anabelle’s repeated partitioning expressed as ‘bams’] Yes but they all … [Perhaps Craig is trying to add the constraint of equal spacing, but this is unclear.]
ANABELLE: [Interrupts Craig, arguing for the arithmetic approach over the geometric] But then that would be more confusing than just, like, timesing it.

In this excerpt, Anabelle and Craig constructed a sequence of shifting microge-
ideas as he drew upon aspects of the presentations that he had just observed. Mathematical ideas travel as students make sense of them.

**Concluding remarks**

Developing methods to understand the travel of ideas is foundational to understanding learning in classroom communities, and we have argued that the genetic perspective on micro-, onto- and sociogenetic processes provides a useful methodological frame. But our approach is preliminary, and we hope that this chapter engenders productive conversation about methodological frameworks and empirical techniques for the study of learning in classrooms.

We are currently expanding the lesson series and conducting iterative cycles of classroom, interview and tutorial studies to guide refinement of lessons and support materials for teachers. In turn, the process of iterative refinement also serves as a laboratory for us as we refine our methodological approach and develop new empirical techniques.

**References**


he or she makes sense of other students' communications in relation to their own ideas. Damian cited Anabelle and Sienna as sources of influence on his thinking, and he did in fact shift to arithmetic and geometric procedures after their presentations and the small-group discussions. But Damian's use of arithmetic and geometric forms did not reflect the functions used by Anabelle or Sienna; he repartitioned only one portion of the number line (between 0 and 1/3), and he multiplied two values (1/3 and 1/8) that were different from those used by either of the presenters. We do not view Damian as misunderstanding Anabelle or Sienna (nor do we view Anabelle or Sienna as being insufficiently clear). Rather, we view Damian's solutions as microgenetic reconstruction of his own earlier

