Relations Between Classroom Practices and Student Learning in the Domain of Fractions

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In this article, we present an analysis of relations between (a) student achievement in the domain of fractions and (b) the extent to which classroom practices are aligned with principles recommended by current reform frameworks (e.g., National Council of Teachers of Mathematics, 1989). Hierarchical linear model analyses were performed on classroom observation and pre- and postinstruction achievement data collected in 19 upper elementary classrooms. These analyses revealed that alignment of classroom practices with reform principles was related to student achievement in problem solving but not in computation; furthermore, the relation differed for students who began instruction with different levels of prior knowledge as indexed by a pretest measure. For students who began with a rudimentary understanding of fractions, the relation between measures of classroom practice and problem solving was linear. In contrast, for students who began without a rudimentary understanding of fractions, the relation was nonlinear; in classrooms rated low on alignment with reform principles, performance on problem-solving items was near floor but increased at a certain threshold level of alignment. The findings demonstrate the value of reform principles as a guide for effective practice as well as the importance of a coordinated analysis of students' prior understandings and classroom practices in investigations of children's learning.

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Reform documents in mathematics education recommend the replacement of curriculum that supports drill and memorization of mathematical procedures with curriculum that supports students’ engagement with conceptual issues in problem solving (e.g., California State Department of Education, 1992; National Council of Teachers of Mathematics, 1989, 1991; National Research Council, 1989, 1990). An emerging body of empirical findings and argumentation provides some support for these recommendations. For instance, empirical studies show that many students acquire procedures—recipes for step-by-step solutions to mathematical problems—without understanding their conceptual rationale (e.g., Brown & Burton, 1978; Brown & VanLehn, 1980; Cauley, 1988). Furthermore, various conceptual treatments of students’ mathematics learning in classrooms point to the importance of student engagement in sense-making activities for the development of mathematical insight and understanding (e.g., see Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Cobb, Wood, & Yackel, 1993; Hiebert & Wearne, 1993; Piaget, 1973). We have yet to develop, however, a strong corpus of studies that examine relations between student learning and the extent to which pedagogical approaches are aligned with reform frameworks. This study is one of the few to explore this relation.

We focus on classroom practices and student achievement in the domain of fractions. In each of 19 elementary classrooms, we classified students into one of two groups based on their performance on fraction problems administered prior to participation in curriculum units involving fractions: those with and those without a rudimentary knowledge of fractions. We anticipated that students who began instruction with a rudimentary understanding of fractions would profit differently from instruction than those students without such understandings. To test our expectations, we analyzed each group’s advances in problem solving and in computation with fractions as a function of the extent to which classroom practices were aligned with reform principles.

FRAMEWORK

We frame our study with assumptions that are common in constructivist treatments of cognitive development (Saxe, 1991; Saxe, Dawson, Fall, & Howard, 1996; Vygotsky, 1978, 1934/1986): Children develop concepts and procedures as they construct mathematical goals and work to accomplish them. Within the constraints of their understandings, students create goals that emerge in and are supported by their participation in classroom practices, goals that they would not create on their own. In students’ efforts to structure and accomplish emergent mathematical goals linked to their own understandings and classroom practices, students create possibilities for generating new learning keyed to instruction.

In the sections that follow, we consider how students’ prior understandings are implicated in their construction of mathematical goals as they address problems
involving fractions. We also consider how classroom practices may differentially support children's emergent goals. The discussion sets the stage for our empirical study of the interplay between students' prior understandings and classroom practices in the process of mathematics learning.

**STUDENTS' DEVELOPING UNDERSTANDINGS OF FRACTIONS**

During the early elementary school years, children shift in their understanding of elementary problems involving fractions (Ball, 1993; Post, Cramer, Behr, Lesh, & Harel, 1993). Various researchers argue that the shift reveals a conceptual transition in children's ability to coordinate relations between parts and wholes (Mack, 1995; Parrat-Dayan & Vonèche, 1992). The shift is revealed when children are asked to represent a fractional part of a larger whole. Consider children's performance on tasks in which they are presented with shapes depicting discontinuous or continuous quantities, such as those contained in Figure 1.

A.

What fraction of the cards is gray? ______

[Diagram of cards]

What fraction of the marbles is gray? ______

[Diagram of marbles]

B.

For each picture below, write a fraction to show what part is gray:

[Diagrams of pie charts]

**FIGURE 1** (a) Items involving fractions for discontinuous quantities and (b) continuous quantities.
Asked by an interviewer, "Write the fraction that shows the amount of the shape that is gray," younger children tend to interpret the goal in terms of whole numbers rather than in terms of part–whole relations. Thus, for the two discontinuous quantity items in Figure 1a, children may count the grays and then indicate the product of their count with a numeral, writing "2" for the first item and "3" for the second item. For the four continuous quantity items in Figure 1b, children may write "3," "1," "4," and "2," respectively. Such whole-number conceptualizations of fractions are not limited to discontinuous and continuous quantity representation tasks. Indeed, in computational activities with fractions, many children will create goals of adding and subtracting numerators and denominators as if they were whole numbers (Gelman, 1991). Similarly, when children are asked to create fair shares—for example, to divide a number of brownies equally among a given number of people—some children will treat the results of their partitions as a number of pieces and not as fractional parts of a whole (Ball, 1993). Thus, to share two brownies among four people, a child might partition each brownie into halves and indicate that each person should receive "one piece."

Students who understand fraction problems in whole-number rather than part–whole terms on elementary fraction tasks such as those depicted in Figures 1a and 1b are likely to profit differently from the same classroom activities involving fractions. Indeed, students with whole-number orientations—that is, without a rudimentary understanding of fractions—should understand classroom lessons and activities about fractions in such terms (or become quite confused by instruction). In contrast, for students with a rudimentary understanding (coordinating part–whole relations on elementary tasks), fraction problems involving arithmetic, fair sharing, or complex representations are more likely to be understood as problems that require a coordination of relations between parts and wholes. Such problem conceptualizations would support children's efforts to address and work through the challenges of introductory lessons about fractions.

CLASSROOM PRACTICES

Children's goals take form not only in relation to their prior understandings but also in relation to the structure of classroom routines, the representational forms that are valued in mathematical lessons, and the patterns of social interaction both between teachers and students and among students themselves (Saxe, 1991). Classroom practice may thus afford children opportunities to structure more complex goals than they create on their own as well as support children's construction of mathematical means to accomplish those goals. To illustrate, consider the goals that children might construct in three different classrooms (i.e., Classrooms A, B, or C) as they engage with a fair-share problem such as the following: "Six people will share three brownies. How much will each person get if each gets a fair share?"
In Classroom A, practices afford students the opportunities targeted by recent reforms. The teacher engages students with approaches to the fair-share problem (six people and three brownies) in ways that build on their understandings, asking students to create solutions to the problem and then monitoring the mathematics that emerge from their efforts. Later, the teacher orchestrates students in discussion of the values of several different solutions, such as $\frac{1}{2}$, $\frac{1}{6}$, “1 piece,” or “6 pieces,” supporting children’s efforts to make sense of pieces as fractional parts of wholes. For example, the teacher asks students to consider whether each solution is a fair share and to explain their answer; when students have produced the “same answer” but represent it differently—for example, drawing “halves” of markedly different sizes—the teacher asks whether the solutions are equivalent. The teacher also supports children’s efforts to reconcile the tension between a “whole brownie” and the “whole set of brownies.” For example, the teacher asks students to analyze why two students might represent the same drawing in two different ways, one as $\frac{1}{2}$ and another as $\frac{1}{6}$. In Classroom A, students with rudimentary insights into fractions may be encouraged to extend their knowledge to new fair-share problems, such as new shapes to partition (e.g., circles instead of squares) or different values (e.g., thirds instead of halves).

In two other classrooms, the practices are less likely to afford students the opportunity to interpret the goal of the fair-share problem in terms of part–whole relations. In one case, the teacher values a procedural approach to problem solving, whereas in the other, the teacher values “student discovery.”

In Classroom B, the teacher draws three rectangles on a blackboard, partitions each into two parts, and tells the students that each of the six people should receive one part and that “we call each part ‘one half.’” Students with a rudimentary understanding of fractions may understand the meanings of the teacher’s activities in terms of fractions; the “whole” is one brownie, and “half of the whole” is one brownie partitioned into one of two equal-sized parts. Building on additional insights gained from this lesson, these students may interpret new fair-share problems in terms of fractions even if the problems vary in representational forms or in values. However, students without a rudimentary understanding of part–whole relations may interpret “one half” as a particular shape or as “one of two pieces of any size.” Such students may be more likely to make sense of the parts as pieces, without an understanding of how pieces may be conceptualized as areas of larger wholes. Thus, these students are less likely to interpret the goal of a fair-share problem in terms of fractions.

In Classroom C, the teacher poses the fair-share problem to students, provides them with manipulative resources for exploring possible methods for solving the problem, and celebrates student work by posting it on the classroom bulletin board. There is little discussion other than student presentations of their work. The insights that any of the students construct emerge on the children’s initiative, the resources at their tables, and the interactions among tablemates as well as the chil-
dren's prior understandings. Students with a rudimentary understanding of fractions may explore the fair-share problem, extending their burgeoning understanding of fractions as relations between parts and wholes. However, students without a burgeoning understanding of part–whole relations are likely to interpret the problem in whole-number terms, profiting little from the activity.

ANALYZING PRACTICES AND STUDENTS' ACHIEVEMENTS

The purpose of our study was to analyze the relation between student learning in the domain of fractions and the extent to which classroom practices were aligned with reform principles, differentiating students who did and did not demonstrate a rudimentary understanding of fractions at pretest. We designed our study to enhance variation in classroom practices in two ways: First, we recruited teachers who were either utilizing a curriculum unit that was intended to support reforms, Seeing Fractions (Corwin, Russell, & Tierney, 1990), or who were using a traditional textbook. Second, for teachers using the reform unit, we provided one of two different professional support programs (described in Gearhart et al., in press). We then followed these classrooms throughout fraction instruction, videotaping and collecting field notes on classroom practices, and assessing children's pre- and posttest performances on items that involved computation and problem solving with fractions.

Rating Classroom Practices

To evaluate the extent to which classroom practices were aligned with reform principles, we developed rating scales and applied them both to videotape and field note records of whole-class lessons (see the Method section). The scales were used to evaluate core instructional principles espoused in reform documents by (a) the degree to which classroom practices elicit and build on students' thinking (integrated assessment) and (b) the extent to which conceptual issues are addressed in treatments of problem solving (conceptual issues). To apply the integrated assessment scale, raters were instructed to attend to teacher questioning and public problem solving and the ways that these did or did not elicit and address students' mathematical understandings. To apply the conceptual issues scale, raters focused on the ways that methods for solving fraction problems were linked to core fraction concepts: part–whole relations, part–part relations, and equivalence relations. Parallel scales were developed for videotape and for field notes, resulting in four scales in all.

Teacher A, based on the previous brief sketch of the classrooms, would receive a high rating on both scales. She elicited students' understandings (integrated as-
essment) by asking students to create and explain solutions to an open-ended problem and then monitored the mathematics that emerged from their efforts. She then built a discussion on students’ thinking—focusing students on their quite different methods—and on conceptual issues related to the various strategies and solutions (conceptual issues).

In contrast, Teachers B and C would receive a low score on both scales. Although these teachers organized instruction in quite different ways, neither made efforts to elicit and sustain discussion of students’ mathematical thinking (integrated assessment). Indeed, Teacher B, who favored a didactic approach, assigned students a procedure for solving division problems involving fractions, explaining what the resulting fractional quantities should be called. Teacher C encouraged students to create their own solutions but did not make efforts to understand or monitor the mathematics in these solutions. Neither teacher supported students’ engagement with conceptual issues involving mathematics (conceptual issues). For students in Teacher B’s class, engagement with mathematics involved memorization and rehearsal of procedures. For students in Teacher C’s class, engagement took a wide range of forms, including projects that some students accomplished with little mathematical thinking.

Assessing Students' Problem Solving and Computation With Fractions

Our assessments of student achievement in the domain of fractions were designed to measure both students’ computational skills and their competence with problem solving. The distinction between computation and problem solving is captured in similar ways by other researchers using such constructs as procedural versus conceptual knowledge (Greeno, Riley, & Gelman, 1984; Hiebert & Lefevre, 1986), syntax versus semantics of mathematics (Resnick, 1982), and skills versus principles (Gelman & Gallistel, 1978). We recognized that the distinction between computation and problem solving would become problematic when we operationalized it as distinctive sets of items. Indeed, a child might solve what we regarded as a computation task using an invented problem-solving strategy or might solve what we classify as a problem using a memorized procedure. Nonetheless, the items that we constructed provided a heuristically useful way to measure students’ skills and problem solving with fractions. The computation items could be solved using routine algorithmic procedures or commonly memorized facts. The problem-solving items could not easily be solved by standard computational approaches and were more likely to require insight into the concepts underlying representations of fractions. In addition to the face validity of the distinction, we validated the distinction between computations and problem solving through confirmatory factor analytic techniques (Saxe & Gearhart, 1998).
Assessing Students' Rudimentary Understanding of Fractions

To partition students into those who demonstrated a rudimentary understanding of fractions and those who did not, we coded students' performance on an additional set of elementary items that were included as part of the pretest. These additional items were elementary fraction problems depicted in Figure 1; one subset involved discontinuous quantity (items in Figure 1a), and the other subset involved a continuous quantity (items in Figure 1b). Consistency of adequate performance on at least one subset was required for children to be regarded as displaying a rudimentary understanding of fractions (see the Method section).

Expected Relations Between Practice and Achievement

We expected students' performances on the problem-solving and computation scales to vary as a function of students' prior understandings and alignment of classroom practices with reform principles. For the problem-solving scale, we expected that students without rudimentary understandings of fractions would be at risk for not learning from instruction if there were little classroom support for children's conceptual engagement with the subject matter (as indexed by a low level of support on our classroom alignment ratings). These students should be prone to interpret classroom activities involving fractions (e.g., representations, lessons, small-group work) in whole-number terms. Thus, classroom activities at lower levels of support should either be very confusing or systematically misunderstood in terms of whole numbers. However, if these students participated in classroom practices involving fractions that were geared for building on their understandings (as indexed by at least a moderate level of support on our classroom alignment ratings), we might expect to see growth in students fraction concepts and even greater growth at high levels of support.

In contrast, we expected that students with rudimentary understandings at the start of instruction would show a different profile of learning as a function of alignment. These students should be more able to make sense of fraction lessons in terms of part–whole relations even if engagement with fraction concepts was not a focus of instruction (i.e., at low levels of alignment). Furthermore, with greater support for conceptual engagement (increasing levels of alignment on our scale), we expected that these students would show greater gains in their understandings.

For the computation scale, we did not expect to find the same pattern of relations between alignment of practices with reform principles and student performance. Indeed, there is little reason to expect that reform practices would influence directly students' developing competence with computation tasks that are often readily solved through memorization of routine facts and algorithms.
Thus, we expected at best a weak relation between alignment of whole-class lessons with reform principles and students’ computation achievement, regardless of students’ prior understanding of fractions.

METHOD

Teacher Selection

Teacher volunteers were solicited through two mailings to upper elementary teachers. One letter requested applications from teachers engaged with Seeing Fractions (Corwin et al., 1990) and My Travels with Gulliver (Kleiman & Bjork, 1991), and the other letter requested applications from teachers committed to teaching with traditional textbooks. Both groups were informed that the study would contribute to understandings of the roles of curriculum in children’s understandings of fractions, measurement, and scale.

The criteria for participation in the traditional group were (a) espoused commitment to use of textbooks, (b) no experience with reform curriculum, and (c) agreement to use textbooks during the project year. Traditional teachers taught fractions and measurement using their school-approved textbook and workbook materials; these teachers were not provided with a professional development program.

The criteria for participation in either of the two reform groups were (a) experience teaching both units prior to entry into the study, (b) agreement to teach these units during our project year, and (c) agreement to implement (as appropriate in their classrooms) methods discussed in staff development meetings. One of the reform groups—collegial support—was provided opportunities for collegial support and collaboration. The teachers in the second reform group—integrated mathematics assessment—participated in a program focusing on the development of their subject matter knowledge, their knowledge of students’ mathematics and motivation, and methods of assessment and instruction. Each group participated in a kickoff event (2 days for collegial support, 1 week for integrated mathematics assessment) followed by an evening program scheduled throughout the year (monthly for collegial support, biweekly for integrated mathematics assessment). Further information on the criteria for selection and the two professional development programs is available in Gearhart et al. (in press). In this article, we collapsed teachers who participated in these two professional development groups into a single reform group.

Our entire sample consisted of 23 teachers and their classrooms. However, four classrooms were not included in our analyses. There were two sources of the attrition: For two classrooms, our measures of classroom practice were incomplete. We dropped an additional two classrooms from the sample because they had insufficient numbers of students classified as showing a less than a rudimentary under-
standing of fractions. (One classroom contained only one student, and the other contained no students classified in this way.)

Student Participants

The students in the study were in the upper elementary grades (\(Mdn = \text{Grade } 5\)). Language background of the students varied. Our measure of English fluency was the proportion of students in each classroom who were rated a 3 or 4 on a 4-point scale of fluency, derived from the school's categorical assignment as well as teachers' judgments. Because classrooms varied in English fluency, we controlled for these differences statistically in our analyses of relations among prior understandings, classroom practices, and student achievement. Ethnicity of children varied: 64% were Latino, 14% were White, 8% were African American, 7% were Asian, and 7% were classified as other.

Procedure

In the following sections, we describe our measures of (a) the alignment of classroom practices with reform principles, (b) students' rudimentary understandings of fractions, and (c) students' competence with computation and problem solving.

Alignment of classroom practices with reform principles. Classroom observations were collected both as videotapes and field notes for specified Seeing Fractions (Corwin et al., 1990) lessons in the reform classrooms and for lessons covering similar content in traditional classrooms. Our choices of videotapes and field notes were matters of both design and feasibility. Videotaping provided us relatively "raw" data and thus the capacity to identify unexpected patterns in classroom practice. Field notes afforded a graduate student assigned to the classroom for the year opportunity for commentary on patterns of implementation and change over time. (To cross-validate observer commentary in classrooms, one set of field notes was collected in every classroom by Maryl Gearhart.)

Because we found the quality of the videotapes and field notes during independent and cooperative work to be uneven, we focused our coding on whole-class discussion episodes. A whole-class episode was defined as (a) teacher-supervised activity and interaction, (b) whose function was either to prepare students for independent or cooperative work on similar problems or to discuss work that students had completed independently or cooperatively.

Our rating scheme for classroom practices built on multiple subscales (for a validation of the scheme, see Gearhart et al., in press). The scale that we use in this
report is a reduction of four measures derived from two subscales: integrated assessment and conceptual issues. 

Integrated assessment is a 4-point subscale designed to rate the degree to which classroom practices elicit and build on students' thinking (see Table 1). Integrated assessment was defined as the extent of opportunity for students to reveal their understandings, to receive interpretations of their contributions, and to provide interpretations of others' contributions. Raters focused on patterns of questioning and engagement of students in public problem solving. We created two measures using this subscale: a rating of videotapes and a rating of field notes.

Conceptual issues is also a 4-point subscale designed to rate the extent of opportunity for students to consider the mathematical concepts that underlie methods for solving problems (see Table 2). The concepts of relevance to our data were part–whole relations (including the ways that part–whole relations are transformed when two fractional quantities are combined). During development of this scheme, we identified two typical patterns of instruction: *Procedural instruction* (P in Table 2) emphasizes step-by-step problem-solving procedures without attention to mathematical concepts; *discovery instruction* (D in Table 2) encourages student discoveries (e.g., multiple ways to solve a problem) without support for conceptual analysis of those discoveries. Raters were encouraged to attend to these patterns, but most instruction was a mix of these two patterns. Therefore, in the end, the P and D codes included in Table 2 were ignored, and we used only the numeric ratings in our analyses. As with our ratings for integrated assessment, we created two measures using the conceptual issues subscale: The first was a rating of videotapes, and the second, a rating of field notes.

As previously noted, our derived measure was produced by integrating the four measures through a principal components analysis. The derived scale captures the degree to which students had opportunities to engage in discussions that addressed conceptual analysis of problem solving built in part on their mathematical thinking.

*Students' rudimentary knowledge.* Children were assigned to one of two categories based on preassessments. Students were assigned to the with rudimentary understanding category if they passed the majority of items for either a set of discontinuity quantity or a set of continuous quantity tasks involving fractions (the tasks used are contained in Figures 1a and 1b). We set this low criterion in an effort to include any evidence of part–whole coordinations for one task type. Those children who did not meet the criterion were assigned to the without rudimentary understanding category.

*Measuring students' achievement.* Students' competence with computation and problem solving was measured with a paper-and-pencil pre- and posttest.
<table>
<thead>
<tr>
<th>Rating</th>
<th>Description</th>
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<tbody>
<tr>
<td>1. Very little effort to ascertain students' understandings</td>
<td>Questions are rarely asked. Neither the teacher nor other students ask students to explain their reasoning. Students do not work through a problem publicly in ways that could reveal their understandings or strategies.</td>
</tr>
<tr>
<td>2. Limited effort to ascertain students' understandings</td>
<td>Questions provide very limited access to students' understandings; students in this classroom would not expect that the teacher or any student would be invested in understanding their interpretations of the mathematics. The purpose of most questions seems to be either to ensure students' attention, to ensure that a model problem has all of its steps appropriately completed, or both. Questions typically require choosing a solution, and content is typically focused on the solutions to problems, whether the response is oral or public work at the overhead or board. (Solutions may be numeric or graphic representations.)</td>
</tr>
<tr>
<td>3. Some effort to ascertain students' understandings</td>
<td>Questions provide some or inconsistent access to students' understandings. Students in this classroom would expect to be asked some questions requiring at times a more extended response. However, students would not expect that the teacher or any student would be consistently invested in understanding their interpretations of the mathematics, and they would not expect that they could contribute an interpretation of another student's work. The teacher does ask students at times to explain their reasoning or to solve and discuss problems publicly, and thus, some questions may be wh- questions (what, how, why, etc.). However, these questions function in pro forma ways (i.e., the question is asked, but nothing is done with the response). Alternatively, the purpose of the question appears to be to check the student's use of a required procedure, rather than to explore what a student understands of that (or any) method of problem solution.</td>
</tr>
<tr>
<td>4. Ongoing effort to ascertain students' understandings</td>
<td>Questions provide access to students' understandings; students in this classroom would expect that the teacher and students are invested in understanding their interpretations of the mathematics. The teacher asks students to explain their work or their reasoning, and when students explain, the teacher extends the students' responses and pursues exploring what the students' understood. The teacher may (a) make an interpretation of the response, (b) ask students to make an interpretation of the response, and (c) compare the students to another student's strategy, reasoning, and so forth. The teacher and students show evidence of valuing the ways that students approach and understand problems.</td>
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Items contained both typical fraction problems in upper elementary texts and the more open-ended and nonroutine problems that mark reform-oriented curricula. Project staff members administered the test to students in all participating classrooms before and after the intervention. The duration of the test was about 40 min. When appropriate, students used a Spanish translation of the test. We created two subscales to measure students' computation and problem solving with fractions. The computation scale was based on students' solutions to items that permitted correct solutions through routine application of conven-
<table>
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<tr>
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<tr>
<td>1. Not integrated</td>
<td>P: The teacher prescribes and requires a predefined series of steps. These may be listed on the board as reference. The goal appears to be learning to use these steps to solve fraction problems of the targeted type (addition); there is no analysis of these steps that could help students reflect on part–whole relations or on the meaning of the algorithms used for adding fractions (combining quantities). D: The teacher encourages students to discover a way to solve the problem. Every method or solution is found “interesting.” Thus, students construct their own methods, but these are not linked or interpreted within a mathematical context; thus, there is no assurance that children are engaged in a mathematical problem that will enable them to construct more advanced understandings of part–whole relations or methods for combining fractional quantities.</td>
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<tr>
<td>2. Limited integration</td>
<td>P: Only one procedure is used or is “right.” Although the procedure is not explicitly required (at least in this lesson), it does seem that everyone knows “what we’re supposed to do.” The goal appears to be to solve the given problem; there is little analysis of these steps that could help students reflect on part–whole relations or on the meaning of the algorithms used for adding fractions (combining quantities). The goal of the discourse appears to be confirming that students are following the correct procedure. D: The teacher encourages students to discover a way to solve the problem. Students construct their own methods, and there is some discussion of these methods when students “share their strategies.” However, the discussion rarely highlights effectively the relations between strategies and either part–whole or combining quantities. There is, at best, a weak analysis of procedures interpreted within a mathematics context.</td>
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<tr>
<td>3. Some or occasional integration</td>
<td>P: Only one procedure is recommended or “right,” but there is acknowledgment that other approaches are possible. There is also some effort to address part–whole relations (e.g., “6/4 is the same as 1 2/4, and that is good thinking, but this time we are working on a solution expressed as a mixed-fraction only”). The procedure is linked to graphics or manipulative patterns in ways that could help students understand a fraction in terms of part–whole relations or combining fractions as combining two quantities. D: There is an effort at comparison or analysis that addresses part–whole relations or combining quantities in the discussion. However, the purpose of the discourse appears to be to collect multiple paths to correct solutions (e.g., (“so this is another way we can solve this problem”); incorrect solutions are rarely examined, and there is weak analysis of relations among the correct solutions. Thus, students may be left with uncertainty regarding the conceptual distinctiveness or similarity of different procedures.</td>
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<tr>
<td>4. Extensive integration</td>
<td>Procedures are treated as strategies with mathematical importance, and a fundamental goal of instruction appears to be to engage students in formalizing procedures of their own devise, to engage students in developing conceptual understandings of conventional procedures, or both. Analysis of procedure(s) includes analysis of the ways procedures address and can reveal part–whole relations, the combining of fractional quantities, or both. Such analysis generally entails consideration of conceptual relations among possible representations (graphic, numeric, linguistic, etc.) for fractional quantities and operations on fractional quantities.</td>
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*Note.* If raters could not distinguish P and D, they were instructed to use the number code only. P = procedural instruction; D = discovery instruction.
For each picture below, write a fraction to show what part is gray:

![Fractions](image)

**FIGURE 2** Items used to assess students' representations of fractional parts of areas.

tional algorithmic procedures. Six of the items involved computations (e.g., $\frac{3}{10} + \frac{2}{5} = \frac{7}{10}$, $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$), four entailed computation of fraction equivalencies (e.g., $\frac{3}{4} = \frac{6}{8}$, $2\frac{1}{2} = \frac{5}{2}$), and one required expressing values in a pie chart.

The problem-solving scale was based on students' solutions to items that could not be solved through routine application of algorithmic procedures or simple memory of common "math facts" and that were more likely to require insight into the concepts underlying representations of fractions. These items included constructing fractions for unequal parts of wholes (see Figure 2), estimating fractional parts of areas, and fair-share problems (see Figure 3).

A confirmatory factor analysis supported the validity of the two scales as measuring distinct forms of knowledge, and Cronbach alphas (indexes of internal consistency) were moderate to high for both scales. A report of these psychometric analyses is contained in Saxe and Gearhart (1998). In addition to the psychometric validation, we examined test booklets for ancillary marks that might show evidence of students' efforts to translate the computation problems into graphical formats that may be easier to understand (e.g., representation of fractions with geometric shapes or number lines). We found no such evidence. Furthermore, on the problem-solving items, we found no evidence of children's use of algorithmic procedures (problems and solutions represented in computational formats).

**RESULTS**

A typical approach to analyzing data like ours is multiple regression, that is, analyzing student achievement on the problem-solving and computation scales as a function of our classroom alignment measure. We chose not to use this approach. Because subsets of children were in the same classrooms and, thus, instructed by the same teachers, the student achievement outcomes within classrooms could be expected to be correlated, violating a core assumption of regression (intraclass correlation). Using a regression approach would ignore the dependence of student achievement within classrooms; as a consequence, the analysis would proceed under the illusion that we have more information than we really do, resulting in mis-
leadingly small standard errors. To avoid these problems, we chose to use hierarchical linear models (HLMs; Bryk & Raudenbush, 1992).

HLMs are particularly appropriate for analyzing nested data (e.g., students nested within different classrooms) because standard errors are adjusted for the degree of intraclass correlation. In our analyses, we employed HLMs that consisted of two models: a within-classroom (Level 1) model and a between-classroom (Level 2) model. In the Level 1 model, we modeled student posttest performance as a function of two student predictor variables (pretest scores and English proficiency). In the Level 2 model, we modeled class mean posttest scores as a function of classroom characteristics (our index of the alignment of classroom practices with reform principles).

Using HLMs, we conducted two sets of analyses: The first set focused on students in our sample who demonstrated a rudimentary understanding of part–whole

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<tr>
<th>Fair Share Problem: 2</th>
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<th>Fair Share Problem: 5</th>
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<tbody>
<tr>
<td><strong>Pizzas and 4 People.</strong></td>
<td><strong>Pizzas and 3 People.</strong></td>
<td><strong>Chocolate Bars and 6 People.</strong></td>
</tr>
<tr>
<td>Four people are going to share these two pizzas equally.</td>
<td>Three people are going to share these pizzas equally.</td>
<td>Six people are going to share these five chocolate bars equally.</td>
</tr>
<tr>
<td>(a) Color in one person's part.</td>
<td>(a) Color in one person's part.</td>
<td>(a) Color in one person's part.</td>
</tr>
<tr>
<td>(b) Write a fraction that shows how much one person gets.</td>
<td>(b) Write a fraction that shows how much one person gets.</td>
<td>(b) Write a fraction that shows how much one person gets.</td>
</tr>
</tbody>
</table>

**FIGURE 3** Items used to assess students’ solutions to fair-share problems.
relations on the pretest \((n = 313)\). The second set focused on the students in our sample who did not demonstrate a rudimentary understanding \((n = 168)\). For each set, class mean posttest scores for both computation and problem-solving items were modeled as a function of our measure of classroom practice, adjusting for differences among classes in pretest performance and language proficiency. The HLM technique produced precision weighted estimates—those classes that contained larger numbers of observations received more weight in the analysis. For instance, those classrooms with low numbers of students without a rudimentary understanding were downweighted, and those with greater numbers received more weight.

**Relations Between Alignment and Problem-Solving Scale**

Figures 4 and 5 contain plots of posttest performances on the problem-solving scale as a function of our measure of the alignment of classroom practices with reform principles (standard scores). The posttest means are statistically adjusted for language background and pretest performance. Visual inspection of the plots for students with and without a rudimentary understanding reveals that both slopes show a positive relation between posttest performance and classroom alignment. However, the character of the slopes differs.

For students with a rudimentary understanding, the relation between posttest performance and our measure of alignment appears linear (Figure 4). Our HLM analyses confirm this. For every unit increase in the classroom practice scale (a 4-point scale), there is a .87 increase in classroom posttest performance (a 13-point scale); the \(t\) statistic shows that the effect is significant, \(t(19) = 4.91, p = .0000\). We interpret these findings to mean that, for children with a rudimentary understanding of fractions, alignment of practice with reform principles is a strong predictor of student learning on the problem-solving items.

For students without a rudimentary understanding, the relation between posttest performance and our measure of alignment does not appear linear (Figure 5). Indeed, for classrooms in which alignment with reform principles was below the mean, the plot appears to show no relation between posttest performance and alignment. In contrast, for classrooms in which alignment with reform principles was above the mean, the plot appears to show a linear relation between alignment and posttest performance. To confirm the visual analysis of the plot in Figure 5, we fit a two-relation HLM to the data. Our model allows for the possibility that the relation between class mean posttest scores and classroom practices may differ for those classes in which alignment is below average and for those classes in which alignment is above average. The results of our HLM analyses supported the two-relation model. When alignment of classroom practices with reform principles is below average, we find no relation between posttest score and alignment, \(t(16) = 0.81, p = .433\). In contrast, when alignment is above average, we find a sig-
FIGURE 4 Adjusted classroom posttest means on the problem-solving scale for students with rudimentary understanding as a function of classroom alignment measure.

significant effect. For every point increase in classroom alignment, there is an expected posttest score increase of 2.07 points, t(16) = 2.48, p = .025. This is a significant relation that is almost 5 times the magnitude of the estimated effect for the relation for below average alignment classrooms.

One interpretation of classroom performances on the problem-solving scale is that performances reflect familiarity with problem formats: Children in classrooms implementing reform curriculum (using conceptually oriented problems as a staple of instruction) tended to achieve greater scores than classrooms that were implementing texts simply because they were familiar with the formats of the test items. Because traditional classrooms tend to score lower on the problem-solving scale than those implementing the reform curriculum, there is merit to this interpretation. There are, however, several features of our data that indicate such familiarity with problem format provides only a limited explanation of student performance.

First, we found considerable variation in the adjusted posttest scores on the problem-solving scale across classrooms implementing the reform curriculum,
FIGURE 5 Adjusted classroom posttest means on the problem-solving scale for students without rudimentary understanding as a function of classroom alignment measure.

both for students with rudimentary understandings (Figure 4) and for students without rudimentary understandings (Figures 5). The reform curriculum, as an index of familiarity with problem format on the problem-solving scale, thus leaves much variation unaccounted for. Moreover, considerable variation was accounted for by our measure of alignment in reform classrooms on the problem-solving scale, indicating that reform principles are important predictors of achievement beyond problem format.

Second, a few of the traditional classrooms achieved posttest scores that were no different from or greater than those of classrooms implementing the reform curriculum. This finding indicates that even traditional curriculum, in particular cases, may lead to greater achievement than reform curriculum on the problem-solving scale.

Third, within the same classrooms, students with rudimentary understandings performed better than students without rudimentary understandings, indicating that prior understanding of fractions, beyond particular problem format, plays a role in student achievement.
Together, these considerations point to the importance of the coordinated analysis of students' rudimentary understandings, curriculum, and classroom practices in student achievement on the problem-solving scale.

Relations Between Alignment and Computation Scale

Figures 6 and 7 contain plots of adjusted posttest performances on the computation items as a function of our alignment measure. In contrast to the problem-solving items, the slopes for students with a rudimentary understanding (Figure 6) and without a rudimentary understanding (Figure 7) reveal no relation between posttest performance and alignment of classroom practice with reform principles. Our HLM analyses confirm these visual observations. For students with a rudimentary understanding, the estimated effect was $-0.26$, $t(19) = -0.70$, $p = .49$; for students without a rudimentary understanding, the estimated effect was also $-0.26$, $t(19) = -0.57$, $p = .58$.

![Graph](image_url)

**FIGURE 6** Adjusted classroom posttest means on the computation scale for students with rudimentary understanding as a function of classroom alignment measure.
FIGURE 7  Adjusted classroom posttest means on the computation scale for students without rudimentary understanding as a function of classroom alignment measure.

The lack of relation between student performance on the computation scale and alignment of classroom practice with reform principles was expected. In the short term, neither support for students’ conceptual engagement with mathematics nor efforts to build on student understanding are likely to enhance students’ memorization of arithmetic procedures. Although some students may be able to extend their developing understandings of fractions to computational items, it may well be that rehearsal of computational procedures under direct instructional methods is more successful in enhancing computational skills. This latter conjecture was beyond the purpose and scope of our analyses.

DISCUSSION

We began our research with assumptions grounded in a constructivist treatment of cognitive development (cf. Saxe, 1991). Students’ mathematics learning is intrinsi-
cally linked to the mathematical goals that students create in their activities; goals that are enabled and constrained by students’ prior understandings. Such goals take varying forms as students make sense of instructional materials, representations, and classroom interactions. In their efforts to create and structure means to accomplish their mathematical goals, students create the possibility of generating new understandings.

In our study, we followed students who began with and without rudimentary understanding of fractions. In our analyses of each group of students, we focused on the relation between classroom practices and students’ mathematics achievement in the domain of fractions as indexed by performance on computation and problem-solving items. We built our analyses of classroom practices on instructional principles valued in current reform documents—the extent to which discussions provide opportunities for conceptual analysis of problem solving built on student understandings. In the following subsections, we offer an interpretation of our findings as we examine relations among students’ prior understandings, classroom practices, and student learning.

Problem-Solving Scale

Student gains on the fraction problem-solving scale were related to alignment of classroom practices with the principles of reform. However, the relation differed for students without and with rudimentary understanding of fractions, a difference particularly evident when we compare classrooms towards the lower (−2 to 0) and upper (0 to +2) ranges of our alignment scale (see Figures 4 and 5).

At low levels of alignment, posttest scores for students without rudimentary understandings were unrelated to increases in alignment of practices with reform principles. We interpret this finding as evidence that, in classrooms judged low on alignment (whether using traditional or reform curricula), students without a rudimentary understanding had little basis on which to structure mathematical goals in other than whole-number or procedural terms. In contrast, at higher levels of alignment, posttest scores for students without rudimentary understandings were related to alignment, and indeed those scores increased sharply. We interpret this pattern as evidence of a threshold of support needed by such children. With such support, students may become engaged with mathematical goals involving fractions, leading to gains in their understanding of fractions.

For students with rudimentary understandings, we find a different profile. Improvement in posttest performance was associated with an increase in alignment even at the low end of the scale. We interpret this pattern as evidence that these students’ rudimentary understandings of fractions allowed them to make sense of fraction problems in part–whole terms even when classroom practices were relatively inconsistent with the principles of reform.
Computation Scale

We found no evidence that the posttest performance of students with or without a rudimentary understanding on the computation items was related to increasing alignment of classroom practices with reform principles. A plausible explanation for this lack of relation between learning and alignment is that the computation items afforded solutions with conventional algorithms that are not well supported with inquiry approaches to instruction. The variation evident in the scatterplots for posttest performances on the computation scale may be related to the extent to which teachers were implementing traditional pedagogical practices that value drill and practice with procedures. However, we did not develop measures to capture traditional pedagogy.

CONCLUSIONS

Understanding relations between classroom practices and student mathematics learning is critical for improving the quality of mathematics education in schools today. Our efforts here are best cast as a beginning in this complex analytic task. We used the construct of students’ emergent mathematical goals to coordinate analyses of students’ prior understandings, curriculum, emergent classroom practices, and student learning. The results demonstrate the promise of our methods as well as the importance of classroom practices that build on students’ thinking and support students’ engagement with conceptual issues in problem solving.

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REFERENCES


