Coordinating Numeric and Linear Units:
Elementary Students’ Strategies for Locating Whole Numbers on the Number Line

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Abstract

Two investigations of 5th graders’ strategies for locating whole numbers on number lines revealed patterns in students’ coordination of numeric and linear units. In Study 1, we investigated the effects of context on students’ placements of 3 numbers on an open number line. For one group (n=24), the line was presented in a thematic context as a ‘race course,’ and, for a second group (n=24), the line was presented as a conventional number line. Most students in both groups placed consecutive whole numbers at appropriate linear distances, but the thematic context group was more likely to place non-consecutive whole numbers at appropriate linear distances. In Study 2 (n=24), students placed numbers on lines marked with two numbers. Most students placed a third number appropriately when the marked numbers were consecutive whole numbers, but not when the labeled numbers were non-consecutive whole numbers. The findings reveal fifth graders’ conceptual difficulties in coordinating numeric and linear units on the number line and a thematic context that can support this coordination.
Coordinating Numeric and Linear Units:

Elementary Students’ Strategies for Locating Whole Numbers on the Number Line

Mathematics educators have recommended systematic use of the number line in the elementary grades (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices, 2010; National Mathematics Advisory Panel, 2008; National Research Council, 2001; Siegler, Carpenter, Fennell, et al., 2010; Wu, 2008). While elementary students do work with number lines in some contexts, educators have expressed concern that curriculum materials provide limited support for student learning about the number line and linear measurement models of number (Barrett & Clements, 2003; Kamii & Clark, 1997; Lewis, Perry, Friedkin & Baker, 2010; Males, Smith, Lee, 2013; Smith, Sisman, Figueras, Lee, Dietiker, & Lehrer, 2008). To design more effective number line curricula, educators need evidence of the prior knowledge that students will build on and transform as they reason about properties of number lines and solve problems involving the representation of numbers on the line. The studies reported here were conducted to further our understanding of upper elementary students’ reasoning in preparation for the design of new number line curriculum for 4th and 5th grades. We targeted the upper elementary grades for research and development, because students’ robust understandings of the number line become essential as students transition to middle school and begin work with representations of functions on the coordinate plane.

Understanding the Number Line: Coordinating Numeric and Linear Units

The number line is a geometric interpretation of number, a line subdivided into intervals that are labeled from left to right with numbers of increasing value. As we review below, reasoning about numbers and reasoning about linear geometric relations develop independently of reasoning about number line representations (see Figure 1); at the same time, students’ developing numeric and geometric understandings provide key resources for their interpretation of number lines. To date research has not focused on the character of students’ developing coordination of numeric and linear units as they interpret and generate number lines, and the studies we report here address this gap in the literature.

Figure 1. The number line as a hybrid representation involving a coordination of linear and numeric units.

<table>
<thead>
<tr>
<th>Numeric units</th>
<th>Linear units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 = 0 + 1,</td>
<td>Coordinating Numerical and Linear Units</td>
</tr>
<tr>
<td>2 = 1 + 1,</td>
<td>1 1 1 1 1</td>
</tr>
<tr>
<td>3 = 2 + 1, ...</td>
<td>0 1 2 3 4</td>
</tr>
</tbody>
</table>

**Students’ developing understanding of numeric units.** While understandings of numeric units develop outside the number line context, these understandings are critical resources for the interpretation of number lines, as depicted in the first column of Figure 1. Conceptualization of the progressive accumulation of discrete elements, each taken once and only once, is fundamental to the idea of numeric unit. Any number can be understood as the composition or decomposition of other numbers (e.g., 3 = 4 - 1 and 2 = 1 + 1).
Evidence of students’ developing understanding of numeric units comes from studies of students’ treatment of discontinuous quantities. Young children begin to construct understandings of numeric units at an early age (Wynn, 1992, 1993), though there is disagreement about how early (Wakeley, Rivera, & Langer, 2000; Wynn, 2000). When searching for objects, toddlers will display expectations that reveal their understanding of addition and subtraction of 1 even without counting (Starkey, 1992). In the preschool years, children often use counting words to enumerate small sets and come to use the last number word of their counts to represent a group of objects (Fuson, 1988; Gelman & Gallistel, 1978; Saxe, 1977; Saxe, Guberman, & Gearhart, 1987; Schaeffer, Eggleston, & Scott, 1974). By the primary grades, students begin to use counting strategies to compose units and collections of units (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Ginsburg, Klein, & Starkey, 1998). These strategies initially involve counting each term of an arithmetic problem by ones (‘counting all’ strategy) and later become more abbreviated – for example, ‘counting on’ from one value to the next (Fuson, 1988). Steffe and his colleagues (Steffe, 1991; Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983) present an analysis of shifts in types of counting that they characterize as a developmental process of progressively more sophisticated constructions and coordinations of numeric units. In their treatment, built upon earlier work by Piaget (1970, 2001), the developmental process begins with pre-numeric counting schemes linked to counting situations, and, through a constructive process of reflective abstraction, builds towards the creation of generalized numeric units.

Students’ developing understanding of linear units. Like the idea of numeric unit, the idea of geometric units of linear distance is fundamental to number lines, but their development is not inherently linked to the number lines (Figure 1). In their seminal investigation of children’s developing understanding of geometric relations, Piaget, Inhelder, & Szeminska (1960) provided empirical support for the argument that children’s conceptions of distance and length develop from non-metric topological conceptions to metric Euclidean conceptions. The tasks used in their investigations were set in non-number line task contexts – for example, describing distances and spatial relations between landmarks in their school neighborhood, manipulation and reflection of lengths as measures, as well as tasks that examined the idea that length remains invariant over spatial translations and subdivisions. Piaget and colleagues also showed that only by middle childhood did children demonstrate Euclidean conceptions of transitive and associative properties of lengths as well as the conservation of length through subdivisions (Piaget, 1965; Piaget, Inhelder, & Szeminska, 1960; see also Inhelder, Sinclair & Bovet, 1974; Laurendeau & Pinard, 1970). More recent studies have corroborated Piaget’s central findings related to his geometric treatment (see Lehrer, 2003 for a review). Research that bears more directly on number lines comes from recent psychological and cognitive neuroscience studies; these studies show that children may have intuitions about the representation of number as linear magnitude on a line, but the character of this ordering does reflect equidistances between consecutive numbers (Dehaene, 1997; Laski & Siegler, 2007; Siegler & Booth, 2004; Thompson & Siegler, 2010).
Coordination of numeric and linear units on number lines. Students’ early understanding of numeric or counting units applied to discontinuous quantities tends to develop prior to their understanding of distance or length divided into congruent units. Students may initially count tick marks or intervals on the number line or on a linear measurement tool, because they interpret distances as non-metric countable units (counting intervals or tick marks) without consideration of geometric congruence (Mitchell & Horne, 2008). Thus, when numbers are positioned on number lines or rulers in canonical numeric progressions of unit distances (e.g., 0, 1, 2, 3 marked at equivalent unit distances), students’ interpretations of the progression are likely to be correct, because students’ counts along the tick marks match the labeled values. Students’ understandings of the coordination of linear and numeric units may not be revealed by their responses to such canonical number lines or rulers. Similarly, when children are asked to construct a linear arrangement of numbers between xxx.

One might think that using measurement tools supports student understanding, but the research suggests otherwise. Students may find it unproblematic to use rulers that are labeled with consecutive whole numbers in correct order but spaced unevenly (Petitto, 1990), may construct a ruler with ‘unit’ intervals that are not congruent (Horvath & Lehrer, 2000; Koehler & Lehrer, 1999), and may use informal tools (such as fingers or erasers) to count out successive number of placements without constructing and iterating linear units of equal length and value (Inhelder, Sinclair & Bovet, 1974; Lehrer, Jenkins, & Osana, 1998; Nunes, Light, & Mason, 1993).

Research Framework and Purpose

To date research has not focused on students’ developing coordination of numeric and linear units as they interpret and generate number lines. With number lines as our key focus, we report two studies that document students’ strengths and difficulties in coordinating numeric and linear units on number lines, and their use of informal measurement tools in this process. In one study, we investigate thematic contexts (a pretend race course) that may support students’ coordination, because there is evidence from research at the secondary level that a measurement context for the number line can support students’ insights about number line representations (Gerace & Mestre, 1982). We regard our findings as an important resource for the design of curriculum tasks and pedagogical approaches to supporting students’ developing understandings of the coordination of numeric and linear units on the integer number line.

The purpose of Study 1 was to determine how 5th grade students construct and coordinate linear and numeric units as they place numbers on open number lines (lines with no tickmarks labeled). Based on prior research, we expected that, if students were asked to place a consecutive sequence (such as 9, 10, 11), they would likely place the numbers in order from left to right at equidistant intervals as on a canonical number line, but, if they were asked to place a non-consecutive sequence (such as 9, 11, and 12), they might place the numbers in order from left to right without regard for metric relationships between the intervals. We also expected that the accuracy of students’ number placements on the line would be supported by the thematic race course context that affords appreciation of the import of metric distance. The purpose of Study 2 was to investigate students’ interpretation and use of linear units when placing a third number on a ‘fixed’ number line – a number line labeled with two numbers. Task representations included number line tasks typical in elementary textbooks (e.g., given a line marked
with 9, 10, mark 11) and non-canonical number line tasks (e.g., given a line marked with 9, 11, mark 12) that required partitioning into intervals of unequal lengths. We expected that Study 2 would corroborate students’ difficulties coordinating numeric and linear units on open number lines despite the information provided by the marked numbers: Students should show greater accuracy with the canonical tasks (e.g., 9, 10, mark 11), but show conceptual difficulties coordinating linear and numeric units with non-canonical tasks (e.g., 9, 11, mark 12).

**Study 1: Students’ Construction of Linear Units on Open Number Lines in Thematic and Number Line Contexts**

To investigate the ways that students construct and use linear units to place whole numbers on open number lines as well as the role of a thematic context as support for understanding, we assigned fifth graders to one of two groups. One group placed cartoon animals on a blank ‘race course,’ and another placed numbers on open number lines. Each group completed parallel tasks that varied in difficulty. For the ‘consecutive numbers tasks,’ whole numbers were consecutive, and the task could be accomplished by constructing number lines like routine textbook lines with equidistant tick marks. For other tasks, at least two of the numbers were not consecutive (e.g., 9, 10, 13; or 9, 12, 13), and correct solutions required students to identify a unit interval and employ it to position numbers. We analyzed students’ unitizing activity as evidenced in the order of numbers and distances between numbers they marked on the line, as well as students’ use of nonstandard tools (e.g., the cap of a pen or a segment of a finger) and gestures (e.g., ‘hopping’ finger gestures along the line) to regulate their placements of numbers. (We did not provide students with standard measurement tools such as rulers.) Comparison of group performances enabled us to document whether the race course context supported students’ insights about magnitudes and distances and thereby their approach to coordinating linear and numeric units.

**Method**

**Participants.** Fifth graders were selected for the study because our pilot data and prior research revealed that fifth and sixth graders had difficulty with number line concepts (Saxe et al., 2007a). Forty-eight fifth grade participants were drawn from 4 elementary schools in 3 districts in an urban area in Northern California (31 girls and 17 boys). The school populations were ethnically, socio-economically, and linguistically diverse, though schools varied in their patterns of diversity. In two schools, students were of predominately Hispanic/Latino descent (87% and 65%), with substantial proportions classified as English Language Learners (87% and 46%). The third school contained students of predominately African-American descent (88%). The fourth school was heterogeneous (48% White, 31% African-American, 12% Hispanic/Latino, 8% Asian). A substantial percentage of the students at all schools were classified as socio-economically disadvantaged (31-91%). At all schools, students had previous exposure to number lines as a part of school instruction, though these number lines were generally marked with equally partitioned intervals labeled with unit or multi-unit sequences.

All students who returned a guardian consent form and assented to participation were considered for inclusion. Teachers then ranked students for mathematics achievement, and students who were ranked exceptionally low or high (outliers) were
excluded. Because schools varied in the diversity of their student population, in each classroom we created matched pairs of students on mathematics achievement and randomly assigned each member of a matched pair to the Number Line or Race Course groups. In each group, there were more girls than boys, reflecting the distribution of gender in the signed consent and assent forms; the gender distribution was 14:9 (girl:boy) for the Number Line group and 16:7 (girl:boy) for the Race Course group.

**Procedures.** Students were interviewed individually, and all interviews were videotaped. The interviews took approximately 20 minutes and consisted of three parts: Introduction, Block I Tasks, and Block II Tasks.

**Introduction.** The purpose of the introductory phase was to explain the task context. As detailed below, the Race Course tasks provide a thematic context, but the Number Line tasks did not.

**Number line tasks.** The interviewer presented a number line with 0, 1, 2, and 3 labeled (see Figure 2) and asked the student whether he or she had ever seen a number line. The interviewer said, “I’d like to show you some things about this number line,” and explained that this number line has the numbers 0, 1, 2 and 3 marked while pointing to each number. The interviewer also stated that other numbers could go on the number line too.

*Figure 2.* Figure used to introduce the number line tasks.
Race Course tasks (thematic context). The interviewer presented the “race course” with cartoon images of animals running (see Figure 3a) and asked students whether they had ever seen a race. Students were then asked to identify which animal was ahead in this race. The interviewer then presented a second race course with animals running (see Figure 3b) and asked which animal was ahead. The interviewer explained that the animals measure the distance that they have run in units of “wugs,” and the distance between 0 and 1 is called “one wug.” The interviewer asked students to show where 2 wugs was on the race course, and then 3 wugs. During the introduction, students were corrected if any response was incorrect.

Figure 3. Materials used to introduce the race course

(a) Race Course

(b) Measuring Distances on Race Course

Task blocks: Consecutive (Block I) and Partial Consecutive and Nonconsecutive (Block II) Tasks. In each group, students were presented with a series of 2 tasks in Block I and 3 tasks in Block II. Block I consisted of two consecutive numbers tasks: (a) 0, 1, 2, and (b) 5, 6, and 7. The purpose of these tasks was to examine students’ construction and use of a linear unit on the number line when given a triad of numbers that differed by a numeric unit of 1. Block II consisted of three tasks (presented in counterbalanced order) in which either only two of the three whole numbers were consecutive, or none of the three whole numbers was consecutive. The first two task types we termed partial consecutive number tasks: In one type, the first two numbers in the triad were consecutive and the third was not (9, 10, 13), and, in the other type, only the last two numbers in the triad were consecutive (9, 12, 13). We anticipated that consecutive numbers (as opposed to non-consecutive numbers) at the beginning of the triad could provide greater support for students’ coordination of numeric and linear units. For the all non-consecutive numbers task (7, 11, 14) no numbers were consecutive. For this task, a successful solution required students to identify a unit interval in order to coordinate numeric and linear units when placing the numbers on the line.

The procedures for the Number Line and Race Course groups were parallel. Students were presented with two cards (see Figure 4) -- a task card containing three numbers and a blank number line (Figure 4a) or race course (Figure 4b) as appropriate to group. On each card, the number with the least value was located in the right-most position, the number with the greatest value in the top position, and the remaining number in the left-most position. The number cards were identical for both groups, with the exception that, for the Race Course group, the racing animals were positioned
adjacent to the numbers. For both groups, the order of tasks was counter-balanced to control for order effects.

Figure 4. Examples of the number line and race course tasks.

Number line tasks. On the first consecutive numbers task (Block I, Task 1), the interviewer presented the number line and the task card and explained, “Here is a number line with no numbers marked on it. Your job is to put these three numbers on the number line. Try to be as exact as possible about where you put the numbers. You can make any marks you need on the paper to determine where to put the numbers.” When presenting the task card, the interviewer pointed to each number on the card from left to right. After students finished marking the numbers on the line, the interviewer asked, “Tell me how you figured it out.”

For each task, after students finished putting the numbers on the number line, the interviewer asked students to explain how they figured out the answer and asked additional prompts in the following situations:

If the student’s explanation was unclear about why numbers were placed where they were on the number line/race course: How did you know exactly where to put the numbers?

If the student was clearly using a measuring tool but did not explain his or her use of the tool: What were you doing when you were [using the marker/your finger/etc.]?

If the student said “estimate” or “about.” How would you do it if you were asked to do it exactly?

If the student used words for which the meaning was unclear: What do you mean by _____?

If the student appeared to be using some sort of tool to figure out where to put marks on the paper, but did not make marks to indicate how he/she was measuring: You’re welcome to make marks on the paper if that’s helpful. (This prompt was given no more than once during the interview.)

Race Course tasks. On the first consecutive numbers task (Block I, Task 1), the interviewer presented the task card and the number line in Figure 4b and explained, “This card shows the distance in wugs that each animal has run.” The interviewer explained that the card indicates that “the turtle hasn’t started yet and is at 0 wugs, the spider has run 1 wug, and the snail has run 2 wugs.” The task directions were: “Your job is to show the distance that each animal ran as exactly as possible on this new race
course. Try to be as exact as possible. You can make any marks you need on the paper to help determine where to put the animals.” After students finished putting the numbers on the number line, the interviewer gave students a sticker of each animal to place below the appropriate number on the race course. The interviewer then asked students to explain how they figured out the answer, and, as necessary, the interviewer followed up with same follow-up prompts used for the number line tasks.

The procedure for each subsequent task was the same, except that the interviewer began with, “So it’s a new day, and the turtle, snail, and spider are racing again.” The interviewer then explained that “Your job is to show the distance that each animal ran as exactly as possible on this new race course. Try to be as exact as possible. You can make any marks you need on the paper to help determine where to put the animals.”

Coding and reliability. Coding schemes are described for each set of findings in the Results section. The schemes were constructed using a sample of videotaped interviews from each study, and two of the authors double-coded all of the interviews. Coder agreement was calculated to ensure that agreement was satisfactory. Coder agreement was 80% or greater for each scheme (86.4% for strategy use and 84.7% for tool use), and disagreements were resolved through discussion.

Results

On all tasks, regardless of condition, virtually all students ordered the three numbers appropriately with values increasing from left to right. We therefore focused our analysis on students’ strategies for coordinating linear and numeric units when placing numbers, and on students’ uses of improvised tools.

Block I tasks: Consecutive numbers tasks. We measured students’ placements of numbers on the number lines and race courses, and found that, while almost all students ordered the numbers correctly on the Consecutive Numbers Tasks (46 of 48 students on Task A and 46 of 48 on Task B), their placements were somewhat inaccurate. To produce a measure of the appropriateness of students’ placements of the consecutive numbers in the Block I tasks, we used a +/- 20% margin of error to code students’ placements as ‘correct’ or ‘incorrect.’

As shown in Figure 5, in both the Number Line and the Race Course conditions, the majority of students responded correctly to the Consecutive number (Block I) tasks: 37 of 48 students produced an appropriate construction on the 0, 1, 2 task and 33 of 48 students produced an appropriate construction on the 5, 6, 7 task. To compare group performances, we created a total score on Block I tasks by assigning one point for each correct answer for each task type (scores ranged from 0-2). A Mann-Whitney U-Test comparing performances of the Number Line vs. Race Course conditions revealed no effect for condition, \( U (N=48) = 282.5, p = .898 \) (two-tailed).
Figure 5. Number of students in the Number Line and Race Course groups coded as correct on the consecutive number tasks (a) 0, 1, 2 and (b) 5, 6, 7.

Block II tasks: Partial consecutive and all-nonconsecutive numbers tasks. Like for the Block I tasks, we measured students’ placements of numbers on the number lines and race courses on the Block II tasks. In addition, we coded the character of students’ strategies – how they coordinated linear units (units of length) and numeric units on these tasks as well as their use of informal tools in their task solutions. Students’ explanations were used to validate or disambiguate codes assigned based on their actions.

Strategy use codes. Strategy codes described students’ coordination of numeric and linear units. No Unit was coded when there was no evidence that the student constructed this coordination; students placed the values in order and approximately equidistant (regardless of the values of the target numbers) or at relative distances that we could not otherwise interpret. Qualitative strategy was coded when there was evidence that students conceptualized the number series in terms of relative magnitudes, but the relative distances that they constructed were markedly inaccurate, and students’ explanations referred only to relative distance; for example, a student asked to place 9, 10, and 13 marked the 13 at approximately the location for 12, and commented “There are more numbers in here,” gesturing to the line between 10 and 13. Unit strategy was coded when students appropriately coordinated numeric and linear units; they constructed and used a linear unit to place additional numbers on the line, and the relative distances between numbers were appropriate.

Tool use codes. Tool codes described the ways that students improvised either a physical resource or gestures to mediate their placements of numbers on the line. No Tool was coded when there was no evidence that the student was using any kind of physical or gestural resource. A Non-metric Tool was gestural activity, such as pulsing gestures along the line, with no apparent attention to unit interval distance. A Metric Tool use was a rigid object, such as a finger tip or finger joint or a marker cap, used to iterate intervals and locate points along the line.

Students’ strategies and use of tools. The frequency distributions of students’ strategy codes by task for race course and number contexts is presented in Figure 6. The figure reveals that, in the Number Line group, the majority of students used qualitative strategies, evidence that most of these students conceptualized the series of numbers on the line in terms of order relations without coordinating numeric and linear units. For
example, the majority of the students in the number line condition used qualitative strategies, whereas the majority of the students in the race course condition used unit strategies.

*Figure 6.* Strategy types used by the Number Line and Race Course context groups.

To evaluate whether the race course context afforded greater use of unit strategies than did the canonical number line, we created a measure of each student’s use of Unit strategies. We summed students’ use of Unit strategies across the three Block II task types (9, 10, 13; 9, 12, 13; 7, 11, 14). Scores ranged from 0 to 3. A Mann-Whitney U-Test comparing groups revealed a main effect for task context, $U (N=48) = 181.500, p = .006$ (two-tailed). Students in the Race Course group used Unit strategies more frequently ($Mdn = 1, Mean Rank=28.94$) than students in the Number Line group ($Mdn = 0, Mean Rank=20.06$).

To investigate the relation between students’ use of tools and their strategies for coordinating numeric and linear units, we examined the distribution of tool use codes for each strategy type. As indicated in *Figure 7*, students rarely used tools if their strategies were coded No Unit, while students who used Qualitative or Unit strategies varied in their use of tools.
Figure 7. Strategy types by task and by group.

An illustration of the independence of strategy and tool use is shown in Figure 8. One student used a Metric tool to accomplish a Unit strategy for locating 9, 12, and 13 (Figure 8a). She marked an initial tick mark in the middle of the line, and then two tick marks to the left and two tick marks to the right, using the distance between her middle and index finger to check that the tick marks were approximately equidistant. She then used the width of her index finger to create a unit interval, partitioning the initial longer intervals she had created; she labeled the second tick mark with a 1, the third with a 2, and continued up to 20. She then placed the animals on the line. She explained that the line was “easy to split it into 20,” because the half point would be 10 in the middle and then the other “half points” would be 5 and 15; she added that she just tried to make “an even number of them” between her initial tick marks.

Another student, in contrast, used no tool to accomplish the same “unit” strategy (see Figure 8b). The student placed a tick mark toward the left of the number line, and, without any physical implement, marked three more tick marks from left to right. She then labeled these marks with the numbers 9 through 13 and placed the animals at 9, 12 and 13 on the line. In her explanation, she said that she “measured it out as good as I could” and pointed to the spaces and marks she created.
Figure 8. Contrasting productions on a race course condition of (a) a student who made use of a metric tool and (b) a student who did not

![Race Course](image)

(a) Metric tool  (b) No tool

The few students who used a Unit strategy in the Number Line group used either a Metric or Non-Metric Tool, while the students in the Race Course condition who used a Unit Strategy did not all use a tool (see Figure 7). We believe this difference is due simply to the small number of students who actually used a Unit strategy in the Number Line group.

Summary. In Study 1, students created number line intervals by positioning three numbers on an open number line. One group of students placed cartoon animals on a “race course,” and another group placed numbers on blank lines. Virtually all students ordered numbers correctly across tasks, and most students in both groups placed consecutive numbers appropriately (Block I tasks: 1, 2, 3 and 4, 5, 6). However, when placing sets of partial consecutive (Block II tasks: 9, 10, 13; 9, 12, 13) and all non-consecutive numbers (Block II task: 7, 11, 14), students who were placing numbers on race courses were more likely to use appropriate Unit strategies than students who were placing numbers on blank number lines. In both groups, students’ use of informal measurement tools was not associated with the appropriateness of their number placements.

Study 2: Using a Marked Interval to Identify Additional Points on A Number Line

The purpose of Study 2 was to investigate how students used a linear interval (marked with two integers as endpoints) to position a third number on a number line. The task engages students with a fundamental and generative idea about linear units: Once two numbers are specified on a line, the positions of all numbers are fixed, and thus, with the specification of a unit interval length (like the distance between 0 and 1 in Figure 9a) or a multi-unit interval length (like the distance between 2 and 4 in Figure 9b), the positions of all whole numbers (more generally, all real numbers) have one and only one position. In Study 2, some tasks required students to iterate the given interval in order to locate the target number (e.g., given positions of 7 and 9, locate 11), while other tasks required students to partition and iterate the given multi-unit interval length in order to locate the target number (e.g., given the positions of 7 and 9, locate 10). These two task types afforded us a close analysis of students’ strategies for coordinating a number sequence with intervals on the number line. As in Study 1, we also examined students’ use of improvised tools for positioning numbers.
Figure 9. Once the positions of any two numbers are defined on the number line, all other numbers are defined.

![Figure 9](image)

**Method**

**Participants.** Twenty-four fifth grade participants (18 girls and 6 boys) were drawn from the same 4 elementary schools as described in Study 1, and the distribution of students’ achievement levels (based on teacher ratings) was comparable to the students in Study 1. The 18:6 asymmetry in girl-to-boy ratio reflected the distribution of student gender in the signed consent and assent forms.

**Procedures.** Participants were interviewed individually, and all interviews were videotaped. The interviews took approximately 20 minutes. Tasks were administered in two blocks.

**Block I: Marked interval (0,1).** Block I consisted of two baseline tasks in which the unit of (0,1) was defined on the number line (see Figure 10), and the student was required to place a mark for a third number on the line. The (0, 1: 2) unit interval/single unit target task required students to iterate the 0, 1 interval once to locate the target number 2. The (0, 1: 4) unit interval / multiunit target task required students to iterate the 0, 1 interval three times to locate the target number 4.

**Figure 10.** Block I task cards with marked (0,1) interval.

![Figure 10](image)

The tasks were presented as follows: “Here is a number line with [leftmost number marked on the number line] and [rightmost number marked on the number line]. Your job is to put the number [the number written in the top left hand corner] on the number line. Try to be as exact as possible about where you put the number.” For the first task, the interviewer also stated that, “You can make any marks you need on the paper to help determine where to put the number.” After the student marked the number on the number line, the interviewer asked, “Tell me how you figured out where to put the number [target number].” The interviewer followed up with the same additional prompts as in Study 1.

**Block II: Marked interval other than the 0, 1 unit length.** Block II consisted of 6 tasks that provided an interval other than 0, 1 marked on the number line (see Figure 11), and the student was asked to place a third target number. There were three task types, with two tasks each.
Unit interval / multiunit target. The marked interval was a unit interval other than (0, 1), and the correct placement of the target number produced an additional interval that was a multiple of the given unit interval (Figure 11i-a (8, 9: 11) and 13i-b (8, 9: 12)).

Multiunit interval / replication to produce multiunit target. The marked interval was a multiple of a unit interval, and the correct placement of the target number resulted in an interval that was one iteration of the marked multiunit interval (Figure 11 ii-a (7, 9: 11) and Figure 11 ii-b (7, 10: 13)).

Multiunit interval / fractional target. The marked interval was a multiple of a unit interval, and the correct placement of the third target number resulted in an interval that was a fraction of the marked unit interval (Figure 11 iii-a (7, 9: 10) and Figure 11 iii-b (7, 10: 11)). Block II tasks were presented in counterbalanced order using the same procedures as Block I tasks.

Figure 11. Block II tasks.

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<th>(a)</th>
<th>(b)</th>
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<tr>
<td>(ii) Multiunit Interval: Multiple Target</td>
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<td>(iii) Multiunit Interval: Fractional Target</td>
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Coding and reliability. We developed two coding schemes for student strategies based on a sample of videotaped interviews from each study, and two of the authors double-coded the interviews. The unit strategy scheme was used to identify whether students coordinated numeric and linear units to position the third number label on the line, and the tool use scheme was used to identify whether students improvised a tool to help regulate their placement. Coder agreement was calculated to ensure that agreement was satisfactory. Coder agreement was 90% or greater for each scheme (95.3% for strategy use and 91.6% for tool use), and disagreements were resolved through discussion.

Strategy use. Strategy use codes described students’ coordination of numeric and linear units. No Unit was coded when there was no evidence that the student constructed this coordination. For example, for the multiunit interval replication task ii-a (see Figure 11), students iterated the (7, 9) distance twice, and placed the 11 at a point whose value was 13; for the multiunit interval /fractional target task iii-a (see Figure 11), students iterated the (7, 9) interval and placed the 10 at the point whose value was actually 11.
Unit strategy was coded when the student iterated and or partitioned the marked interval in order to place the target number at an approximately appropriate distance from the marked interval. For example, for the multiunit interval replication task ii-a (see Figure 11), students iterated the (7, 9) distance once to locate 11; for the multiunit interval /fractional target task iii-a (see Figure 11), students either partitioned the (7, 9) interval and iterated the unit interval to locate 10, or iterated the (7, 9) interval to locate 11, and then partitioned the (9, 11) interval to locate 10.

**Tool use.** Tool use codes described the ways that students appropriated some material resource to mediate their placements of numbers on the line. *No Tool* was coded when there was no evidence that the student was using any kind of tool. A *Non-metric Tool* was gestural activity, such as pulsing gestures along the line, with no apparent attention to unit interval distance. A *Metric Tool* use was a rigid resource, such as a fingertip or finger joint or a marker cap, used to iterate intervals in order to locate points along the line.

**Results**

We examined students’ strategies as well as relationships between strategy and tool use for Block I and Block II tasks.

**Block I tasks: Marked interval of 0, 1.** As shown in Figure 12, on both the unit interval/single unit target task (0,1: 2) and the unit interval/multiunit target (0,1: 4), almost all students used an appropriate Unit strategy in order to locate the target number. Students’ strategies were not related to their use of tools: Of those students who used unit strategies on the unit interval tasks (almost all), some used no tools, others used non-metric tools, and still others used metric tools.

*Figure 12.* Relationship between strategy and tool use on Block I tasks with (0,1) marked.
Block II tasks: Marked interval of 8,9 or 7,9 or 7,10. The upper part of Figure 13 contains the frequency of Unit strategies on each of the Block II tasks with the associated findings for tool use. We first consider strategy and then the relation between strategy and tool use.

Figure 13. Strategy, tool use, and precision scores across task conditions

To analyze strategy use on Block II tasks, we assigned 1 point if the student used an appropriate Unit strategy to locate the target number on the line for each task type: Unit Interval: Multiunit Target, Multiunit Interval: Multiunit Target, and Multiunit Interval: Fractional Target. There were two tasks for each type, so the scores ranged from 0 to 2 for each item type. Friedman’s ANOVA revealed a statistically significant effect of task type on use of unit strategy (2, n = 24) = 33.552, p < .001). Follow-up pairwise comparisons with Wilcoxon Signed Ranks Test indicated that students were more likely to use an appropriate unit strategy on the unit interval (8,9) with multiunit target tasks (Mdn = 2.0, Mean Rank = 2.75) than on either the multiunit interval with a multiunit target tasks (Mdn = 0, Mean Rank = 1.65), Z(N=24)= -4.001, p < .001 (two-tailed), or the multiunit interval with a fractional target tasks (Mdn = 0, Mean Rank = 1.60), Z (N=24)= -4.066, p < .001 (two-tailed). There was no significant difference in performance on either of the multiunit interval task types, Z (N=24)= -.577 p =.564 (two-tailed).5

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Like Study 1, we found variation in tool use for unit strategies in the Block II tasks, as shown in Figure 13. For all task types, Unit strategies were sometimes accomplished with Metric tools, sometimes Non-metric tools, and sometimes no tools. For the two multiunit interval tasks (see Figure 13b and Figure 13c), No Unit strategies also showed variation in tool use.

To illustrate the independence of strategy and use of tools, consider the work of two students, each of whom used their fingers as a Metric Tool on a multiunit task with multiunit target (7, 9: 11). One student used a non-Unit strategy, iterating the multi-unit distance and treating it as the unit distance: He placed two fingers between 7 and 9 on the line (see a in Figure 14i) and then moved his fingers so that the leftmost finger lined up with the 9 (see b in Figure 14i), adding a tick mark to the right of his fingers, which he later referred to as 10. To complete the placement, he moved his fingers so the leftmost side of his fingers lined up with the newly marked tick mark (see c in Figure 14i), and he finished by creating a new tick mark to the right, labeling it as 11. The second student used a metric tool in the service of a Unit strategy. She began by inserting fingers between the 7 and 9 (labeled ‘a’ in Figure 14ii) and then moved her fingers to the right of the tick mark labeled 9 (see b in Figure 14ii); she created a new tick mark to the right of her fingers and labeled the mark 11. Thus she iterated the multi-unit distance (7 to 9) once to place the 11 at an appropriate location.

*Figure 14.* Contrasting examples of strategies accomplished with a metric tool.

(i) No Unit

(ii) Unit

To provide an index of the accuracy of students’ placements of the target numbers on their number lines for the Block II tasks, we computed precision scores for each of students’ six placements. The precision scores provide a measure of the extent of deviation from the ideal placement. Precision scores were determined by (1) measuring the distance from the rightmost tick mark to a student’s tick mark for the target number (see a, b, c, or d in Figure 15) and (2) dividing the measurement (a, b, c, or d) by the ideal measurement (in Figure 15, the measure of b). The result was a ratio \((\text{measurement } a)/\text{ideal measurement})\) that indicated degree of precision of the
student’s placement. For example, if the ideal value in Figure 15 were 2 cm (where the number 10 should be placed), and the measure for a student’s placement, \( a \), was 1.0 cm, then the precision score would be \( 1\text{cm}/2\text{cm} \) or .5. Thus, regardless of the task, an index of 1.0 indicated a precise placement; 2.0 indicated that the student’s placement was twice as far from the originating mark as the ideal measurement; and .5 indicated that the student’s placement was \( \frac{1}{2} \) as far from the originating tick mark than the ideal measurement.

Figure 15. Computation of precision scores for placements of numbers on the line

![Diagram showing precision scores for different positions of 10](image)

The lower part of Figure 13 contains the bar charts of strategy and tool use across the Block II tasks in the upper part of the figure (Block I is included for comparison), and corresponding box plots for students’ precision scores in the lower part of the figure. The box plots reveal that Unit strategies were associated with a distribution of placements around the 1.0 precision score, with expected minor deviations from the ideal given that all measurements have an element of imprecision. The distributions for the non-unit strategies reveal a pattern of students’ methods of coordinating numeric and linear units. For the multiunit interval:multiunit target tasks ((7, 9: 11) and (7, 10: 13)), and multiunit interval:fractional target tasks ((7, 9: 10) and (7, 9: 11)), students’ placements tended to be overestimates of the precise position by factors of 2 and 3. These ratios indicate that students tended to iterate the given multiunit interval as a linear unit to locate the target number, and, in doing so, they confused a multiunit interval with a unit interval.

Summary. The purpose of Study 2 was to investigate how students used a given marked interval to position a third number on the number line. Students were more likely to place a third number appropriately when presented with a number line labeled with a unit interval (such as 8,9) than when presented with a multiunit interval (such as 7,9). When shown a line labeled with a multiunit interval, students were likely to treat the given interval like a unit interval, and iterate it without appropriately partitioning the interval to identify the linear unit. Students’ use of informal measurement tools was not associated with their strategies for locating numbers.

Discussion

Prior research on students’ developing understanding of linear measurement models of number has been conducted with students at multiple age levels and in varied representational contexts. In the present studies, we extended this work through our focus on the number line, a central mathematical representation used in elementary and secondary curriculum materials. We investigated specific challenges that the number line poses in students’ conceptualization and coordination of numeric and linear units, and our research strategy featured tasks with non-canonical number representations to reveal patterns of student understanding. Because understanding of numeric units tends
to precede understanding of linear units, we expected that students’ solutions would be more accurate for canonical tasks that could be solved based on ordinal numeric relations without attention to metric linear relations. Further, we expected that a thematic race course context might support some students’ coordination of numeric and linear units, given the importance of metric relations in a race. The findings revealed three patterns, and each has implications for further research and practice.

First, our findings indicate that many fifth graders do not have a rich and generative understanding of the coordinated relation of numeric and linear units on the number line. When all numbers were consecutive, students’ placements on both open number lines (Study 1) and fixed number lines (Study 2) were often appropriate. Evidence of gaps in student understanding was revealed by students’ solutions to non-canonical tasks – in Study 1, when placing 3 non-consecutive numbers on an open line (e.g., 9, 11, 12), and, in Study 2, when placing a third number when only two non-consecutive numbers were marked (e.g., numerals 9, 11 marked, place 12). When students positioned three non-consecutive numbers on an open line, their placements did not coordinate linear and numerical units, often inappropriately placing the numbers equidistant from one another. When students placed a third number on a line marked with an interval (Study 2), their ‘errors’ were often the result of using a multiunit interval as the unit interval to locate numbers.

This first pattern in our results provides important information on student understandings that may not be adequately assessed or addressed in curriculum materials or assessments using canonical number lines. Our findings indicate that students’ interpretations and constructions of canonical lines can inflate estimates of students’ generative understanding of number line properties and conventions. Students’ prior opportunities to learn are likely important factors in our results – while number lines do appear in elementary curriculum, these number lines tend to be canonical lines with equally spaced tick marks representing a fixed numeric progression, and lessons rarely provide students opportunity to develop understandings of these representations.

A second pattern in our findings was the role of a thematic measurement context in supporting student’ insights about number lines. Students often placed irregularly sequenced non-consecutive numbers inappropriately on an open number line, but students whose lines were depicted as race courses were more likely to place non-consecutive numbers appropriately (Study 1). While some representational contexts may introduce confusion, our findings indicate that, in this study, a thematic race course context provided students a foothold into complex mathematical ideas of linear measurement and the number line. The idea of a race on a path where linear distance is a core component of what it means to ‘race’ appeared to cue students’ recognition that linear and numeric units must be coordinated to determine how far someone has raced.

The third pattern is related to students’ use of their own improvised tools in solving the number line tasks. Students’ use of informal measurement tools (such as fingers or marker caps) was not linked to their conceptualizations of units as they accomplished our tasks. Our finding makes clear that students’ mere use of tools when solving number line tasks is independent of their developing coordination of linear and numeric units. Indeed, students may use an informal tool to iterate a multiunit interval like a unit interval, and thus place numbers inappropriately, or students with a solid
understanding of the coordination of linear and numeric units may eyeball and place numbers with reasonable accuracy without using an improvised tool.

Our findings inform the questions that future research should investigate to gain additional insight into students’ developing understandings of the number line. First, what other types of thematic contexts can support students’ coordination of numeric and linear units on the number line? Second, what curricular scope and sequence would best support the progressive construction of student understanding of the number line? Studies addressing these questions are important to the design of mathematics curriculum materials that support students’ understandings of representation of integers on the number line.

Concluding Remark

Our research program builds on the tradition of design research in mathematics education, particularly those programs of research that target the learning opportunities afforded by representational contexts. In Realistic Mathematics Education and its extensions, a focus has been on the role of the open number line and linear models in supporting students’ understandings of number and operations (see, for example, Gravemeijer, 2004; Klein Beishuizen, & Treffers, 1998; Selter, 1998). Paul Cobb and his associates have investigated students’ engagement with computer-generated bar graphs and other “designed artifacts” in statistics (Cobb, Confrey, Lehrer, Schauble, 2003; Sfard & McClain, 2002). Researchers in the Cognitively Guided Instruction group have investigated ways that students model number values and operations to support their developing understandings of arithmetic operations (Carpenter & Fennema, 1992; Carpenter, Fennema, & Franke, Levi, & Empson, 1999). In these efforts, researchers have documented the intellectual resources that students build upon as they work with representations and construct mathematical insights, and this developmental foundation has informed the design of curriculum materials and professional supports for teachers. Our own design research program began with investigations of the conceptual resources that students bring to number line tasks (the present studies included) and the ways that students build upon these resources as they generate and evaluate integers and fractions on the number line with the guidance of tutors, teachers, and peers (Saxe, Earnest, Sitabkhan, et al., 2010; Saxe, Gearhart, Shaughnessy, et al. 2009; Saxe & Shaughnessy, 2008; Saxe, Shaughnessy, Shannon, et al., 2007). The findings of this suite of studies have informed the design of a multi-lesson curriculum unit, Learning Mathematics through Representations, on integers and fractions, and the particular contribution of the studies reported here included the design of non-canonical lesson tasks, representations, and the thematic race course context (cf. Gearhart & Saxe, in press; Saxe, Diakow, & Gearhart, in press; Saxe, de Kirby, Sitabkhan, & Kang, in press).
References


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Footnotes

1 The percentages refer to those students whose parents have not received high school diplomas or a student who participates in the free or reduced-price lunch program.

2 If the distance between the second and third target numbers was between 80% and 120% percent of the distance between the first and second target numbers, the answer was coded as correct. For those students whose answers were not coded as correct, we then examined whether the distance between the first and second target numbers was between 80% and 120% of the distance between the second and third target numbers. Answers that met these criteria were also coded as correct. If an answer met neither criterion, it was coded as incorrect.

3 We intended to apply codes for Strategy and Tool Use (see schemes for Block II tasks), but coder agreement was not satisfactory, because students typically completed Block I (consecutive series) tasks rapidly and without obvious reflection. Students either marked the numbers in order from left to right, or placed the smallest and largest numbers at either end of the number line, and then the middle number in the middle of the line.

4 We use italics in our numeric task notation to distinguish the number to be marked from the given numbers on the number line. For example, in the notation, “n₁, n₂: n₃” n₁ and n₂ are marked on a given number line, and the participant is required to mark the position of n₃.

5 We utilized a Bonferroni correction to control for Type 1 error with multiple post-hoc comparisons. We rejected the null hypothesis only if \( p < 0.02 \); this value was computed by dividing .05 by the total number of possible post-hoc comparisons (3).